Vector Multiplication

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Introduction

When dealing with scalars, there is only one definition of multiplication: you multiply one scalar by another scalar, giving a scalar result according to the usual laws of arithmetic: $a \times b = c$. But with vectors, there are *three* different kinds of multiplication: one kind gives a scalar result, another gives a vector result, and another gives a tensor result. Here we'll summarize the various types of vector multiplication and show how to compute each in terms of the rectangular components of the vector.

To begin, let's represent vectors as *column vectors*—that is, 3×1 matrices. We'll define the vectors **A** and **B** as the column vectors

$$\mathbf{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}; \qquad \mathbf{B} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} \tag{1}$$

We'll now see how the three types of vector multiplication are defined in terms of these column vectors and the rules of matrix arithmetic.

Dot Product

The first type of vector multiplication is called the *dot product*. This type of multiplication (written $\mathbf{A} \cdot \mathbf{B}$) multiplies one vector by another and gives a *scalar* result.

The dot product of two vectors **A** and **B** is the product of their magnitudes times the cosine of the angle between them: $\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$. In terms of rectangular components, this is equal to the transpose of column vector **A** times column vector **B**, which gives a 1×1 matrix (i.e. a scalar):

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{A}^T \mathbf{B} = \begin{pmatrix} A_x & A_y & A_z \end{pmatrix} \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = A_x B_x + A_y B_y + A_z B_z. \tag{2}$$

The dot product is commutative ($\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$).

Cross Product

The second type of vector multiplication is called the *cross product*. This type of multiplication (written $\mathbf{A} \times \mathbf{B}$) multiplies one vector by another and gives a another *vector* as the result.

The result of the cross product operation is a vector whose magnitude is $|\mathbf{A} \times \mathbf{B}| = AB\sin\theta$, where θ is the angle between the two vectors. The direction of $\mathbf{A} \times \mathbf{B}$ is perpendicular to the plane containing vectors **A** and **B**, in a right-hand sense.

Unlike the other two kinds of multiplication, the cross product is only defined for *three-dimensional* vectors.

A convenient mnemonic for finding the rectangular components of the cross product is through a matrix determinant:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y B_z - A_z B_y)\mathbf{i} - (A_x B_z - A_z B_x)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k}. \tag{3}$$

Another way to represent the components of the cross product is to write the components of vector **A** into a 3×3 matrix, then multiply that matrix by the column vector **B**:

$$\mathbf{A} \times \mathbf{B} = \begin{pmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{pmatrix} \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} A_y B_z - A_z B_y \\ A_z B_x - A_x B_z \\ A_x B_y - A_y B_x \end{pmatrix}. \tag{4}$$

The cross product is *anti*-commutative ($\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$) and non-associative ($\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$).

Direct Product

The third type of vector multiplication is called the *direct product*, and is written **AB**. Multiplying one vector by another under the direct product gives a *tensor* result. A tensor is a 3×3 matrix that is used to represent certain quantities as stress and pressure.

The rectangular components of the direct product may be found by matrix multiplication: one multiplies the column vector **A** by the transpose of **B**, which gives a 3×3 matrix:

$$\mathbf{AB} = \mathbf{AB}^T = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \begin{pmatrix} B_x & B_y & B_z \end{pmatrix} = \begin{pmatrix} A_x B_x & A_x B_y & A_x B_z \\ A_y B_x & A_y B_y & A_y B_z \\ A_z B_x & A_z B_y & A_z B_z \end{pmatrix}. \tag{5}$$

The direct product is non-commutative ($\mathbf{AB} \neq \mathbf{BA}$).

Vector Product Identities

A few vector product identities are of interest:

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \mathbf{B} \times \mathbf{C} \cdot \mathbf{A} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} = \mathbf{C} \times \mathbf{A} \cdot \mathbf{B} \tag{6}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \tag{7}$$

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) \tag{8}$$

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) \tag{9}$$

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \times \mathbf{B} \cdot \mathbf{D})\mathbf{C} - (\mathbf{A} \times \mathbf{B} \cdot \mathbf{C})\mathbf{D} \tag{10}$$

Note that in the "vector triple product" $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$, there is no ambiguity in the order of operations: the cross product must be done first. (Attempting to do the dot product first results in the cross product of a scalar with a vector, which is not defined.) Eq. (6) says that the order of the dot and cross products may be interchanged, or the order of the vectors permuted cyclically, without changing the result. The result is a scalar whose absolute value is equal to the volume of a parallelepiped defined by the three vectors.

The products in Eqs. (7) and (8) may be summarized as: "The middle vector times the dot product of the two on the ends, minus the dot product of the two vectors straddling the parenthesis times the remaining one."