

# Rolling Bodies

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## Introduction

The motion of a body like a sphere or a cylinder *rolling* (without slipping) down an inclined plane introduces a complication into the motion: the body as a whole moves down the incline, while at the same time the body rotates about its axis. The net movement of the body is a combination of both motions: a *translational* movement of the whole body down the incline, together with a *rotational* motion about its axis. We'll examine here the velocity, acceleration, and kinetic energy of a round body rolling down an incline.

## Velocity

Let's imagine the following scenario: suppose we have an inclined plane, inclined at an angle  $\theta$  to the horizontal. Now place a round body of mass  $M$  and radius  $R$  at a height  $h$  above the base of the incline. If we release the body from rest, what will be its speed  $v$  at the bottom of the incline?

Let's look at the problem from a point of view of energy. At any given instant, the rolling body will be pivoting about the point of contact with the incline (we'll call this point  $P$ ). Its total kinetic energy is therefore the rotational kinetic energy

$$K = \frac{1}{2}I_P\omega^2, \quad (1)$$

where  $I_P$  is the moment of inertia about  $P$  and  $\omega$  is the rotational angular velocity of the body. Now by the parallel axis theorem, we know

$$I_P = I_{\text{cm}} + MR^2, \quad (2)$$

where  $I_{\text{cm}}$  is the moment of inertia of the body about its center of mass. Substituting into Eq. (1), we get

$$K = \frac{1}{2}(I_{\text{cm}} + MR^2)\omega^2 \quad (3)$$

$$= \frac{1}{2}I_{\text{cm}}\omega^2 + \frac{1}{2}MR^2\omega^2. \quad (4)$$

Now using  $v = R\omega$  in the second term on the right, we have

$$K = \frac{1}{2}I_{\text{cm}}\omega^2 + \frac{1}{2}Mv^2. \quad (5)$$

This says that the total kinetic energy of the body is the sum of the rotational kinetic energy (the first term on the right) and the translational kinetic energy (the second term on the right).

Now let's use the conservation of energy to solve for the speed  $v$  at the bottom of the incline. At the top of the incline, the body is at rest, and its energy is all potential and equal to  $Mgh$ . At the bottom of the incline, the energy is all kinetic, and is given by Eq. (5). Then by conservation of energy,

$$Mgh = \frac{1}{2}I_{\text{cm}}\omega^2 + \frac{1}{2}Mv^2. \quad (6)$$

Substituting  $\omega = v/R$  into the first term on the left,

$$Mgh = \frac{1}{2}I_{\text{cm}}\left(\frac{v}{R}\right)^2 + \frac{1}{2}Mv^2. \quad (7)$$

Now out factor  $v^2/2$  on the right-hand side to get

$$Mgh = \left(I_{\text{cm}}\frac{1}{R^2} + M\right)\frac{v^2}{2}. \quad (8)$$

Now dividing through by  $M$ ,

$$gh = \left(\frac{I_{\text{cm}}}{MR^2} + 1\right)\frac{v^2}{2}. \quad (9)$$

The dimensionless combination  $I_{\text{cm}}/(MR^2)$  occurs often enough that it's convenient to introduce the abbreviation

$$\beta \equiv \frac{I_{\text{cm}}}{MR^2}. \quad (10)$$

(Values of  $\beta$  for several common geometries are shown in Table 1.) With this definition, Eq. (9) becomes

$$gh = (\beta + 1)\frac{v^2}{2}. \quad (11)$$

Solving for  $v$ , we finally have the speed at the bottom of the incline given by

$$v = \sqrt{\frac{2gh}{\beta + 1}} \quad (12)$$

## Acceleration

Now let's find the (translational) acceleration of the body down the incline. If the distance down the incline is  $x$ , then the velocity  $v$  at the bottom of the incline is related to  $x$  by

$$v^2 = 2ax \quad (13)$$

By geometry,  $\sin \theta = h/x$ , and so  $x = h/\sin \theta$ ; using this to substitute for  $x$ , we have

$$v^2 = 2a\frac{h}{\sin \theta}, \quad (14)$$

or, solving for the acceleration  $a$ ,

$$a = \frac{v^2 \sin \theta}{2h}. \quad (15)$$

Now let's use Eq. (12) to substitute for  $v$ ; the result is an expression for the acceleration of a body rolling down an incline,

$$\boxed{a = \frac{g \sin \theta}{\beta + 1}} \quad (16)$$

Table 1 shows values of  $\beta$  and  $a$  for several common geometries.

## Kinetic Energy

As a body rolls down an incline, its potential energy is converted partly into translational kinetic energy, and partly into rotational kinetic energy. How much goes into translational kinetic energy, and how much into rotational?

First, let's compute the *translational* kinetic energy,  $K_t = Mv^2/2$ . Using Eq. (12) to substitute for  $v$  gives

$$K_t = \frac{1}{2} M v^2 = \frac{1}{2} M \left( \frac{2gh}{\beta + 1} \right), \quad (17)$$

or

$$\boxed{K_t = \frac{Mgh}{\beta + 1}} \quad (18)$$

Now let's find the *rotational* kinetic energy,  $K_r = I_{\text{cm}}\omega^2/2$ . Using  $\omega = v/R$ ,

$$K_r = \frac{1}{2} I_{\text{cm}} \left( \frac{v}{R} \right)^2. \quad (19)$$

Again using Eq. (12) to substitute for  $v$ ,

$$K_r = \frac{1}{2} \frac{I_{\text{cm}}}{R^2} \frac{2gh}{\beta + 1}. \quad (20)$$

Multiplying the numerator and denominator by  $M$ ,

$$K_r = \frac{I_{\text{cm}}}{MR^2} \frac{Mgh}{\beta + 1}. \quad (21)$$

The first factor on the right is just  $\beta$ , so we finally have for the rotational kinetic energy

$$\boxed{K_r = Mgh \left( \frac{\beta}{\beta + 1} \right) = \beta K_t} \quad (22)$$

Knowing that the total kinetic energy is  $K = Mgh$ , we can now use Eqs. (18) and (22) to find the ratio of the translational kinetic energy to the total kinetic energy:

$$\frac{K_t}{K} = \frac{1}{\beta + 1}. \quad (23)$$

Similarly, the ratio of the rotational to total kinetic energy is given by

$$\frac{K_r}{K} = \frac{\beta}{\beta + 1}. \quad (24)$$

Values of these ratios for common body geometries are shown in Table 1.

Table 1. Accelerations and energy ratios for rolling bodies.

Body	$\beta$	$a$	$K_t/K$	$K_r/K$
Cylindrical shell	1	$(1/2) g \sin \theta$	1/2	1/2
Solid cylinder	1/2	$(2/3) g \sin \theta$	2/3	1/3
Spherical shell	2/3	$(3/5) g \sin \theta$	3/5	2/5
Solid sphere	2/5	$(5/7) g \sin \theta$	5/7	2/7