

# Newton's Method

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## Introduction

As we have seen in the study of projectile motion, some problems in physics result in equations that cannot be solved in closed form, but must be solved numerically. The study of the methods of solving such problems is the field of *numerical analysis*, and is a course in itself. Here we look at one very simple method for numerically finding the roots of equations, called *Newton's method*.

## The Method

Newton's method is a numerical method for finding the root(s)  $x$  of the equation

$$f(x) = 0. \tag{1}$$

The method requires that you first make an initial estimate  $x_0$  of the root. From that initial estimate, you calculate a better estimate using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2}$$

Applying this formula (with  $n = 0$ ) to the initial estimate  $x_0$  gives a better estimate  $x_1$ . This better estimate  $x_1$  is then run through the formula again ( $n = 1$ ) to get an even better estimate  $x_2$ , etc. The process may be repeated indefinitely to yield a solution to whatever accuracy is desired.

If the equation  $f(x) = 0$  has more than one root, then the method will generally find the one closest to the initial estimate. Choosing different initial estimates closer to the other roots will find those other roots.

If there is no root (for example,  $f(x) = x^2 + 1 = 0$ ), the method will tend to "blow up": instead of converging to a solution, you may just get bigger and bigger numbers, or you may get a series of different numbers that show no sign of converging to a single value.

## Example: Square Roots

When the first electronic calculators became available in the mid-1970s, many of them were simple "four-function" calculators that could only add, subtract, multiply, and divide. My dad showed me how I could calculate square roots on one of these calculators using Newton's method, as described here.

To calculate the square root of a number  $k$ , we wish to find the number  $x$  in the equation

$$x = \sqrt{k}. \tag{3}$$

Squaring both sides then subtracting  $k$  from both sides, we get a function of the form of Eq. (1):

$$f(x) = x^2 - k = 0. \quad (4)$$

The values of  $x$  that satisfy this equation are the desired square roots of  $k$ . Newton's method for finding square roots is then Eq. (2) with this  $f(x)$  (and with  $f'(x) = 2x$ ):

$$x_{n+1} = x_n - \frac{x_n^2 - k}{2x_n}. \quad (5)$$

For example, to calculate  $\sqrt{5}$ , set  $k = 5$ . Make an initial estimate of the answer—say  $x_0 = 2$ . Then we calculate several iterations of Newton's method (Eq. 5) to get better and better estimates of  $\sqrt{5}$ :

$$x_0 = 2 \quad (6)$$

$$x_1 = x_0 - \frac{x_0^2 - 5}{2x_0} = 2 - \frac{2^2 - 5}{2 \times 2} = 2.2500 \quad (7)$$

$$x_2 = x_1 - \frac{x_1^2 - 5}{2x_1} = 2.2500 - \frac{2.2500^2 - 5}{2 \times 2.2500} = 2.2361 \quad (8)$$

$$x_3 = x_2 - \frac{x_2^2 - 5}{2x_2} = 2.2361 - \frac{2.2361^2 - 5}{2 \times 2.2361} = 2.2361 \quad (9)$$

After just a few iterations, the solution has converged to four decimal places: we have  $\sqrt{5} = 2.2361$ .

There are actually two square roots of 5. To find the other solution, we choose a different initial estimate—one that is closer to the other root. If we take the initial estimate  $x_0 = -2$ , we get

$$x_0 = -2 \quad (10)$$

$$x_1 = x_0 - \frac{x_0^2 - 5}{2x_0} = -2 - \frac{(-2)^2 - 5}{2 \times (-2)} = -2.2500 \quad (11)$$

$$x_2 = x_1 - \frac{x_1^2 - 5}{2x_1} = -2.2500 - \frac{(-2.2500)^2 - 5}{2 \times (-2.2500)} = -2.2361 \quad (12)$$

$$x_3 = x_2 - \frac{x_2^2 - 5}{2x_2} = -2.2361 - \frac{(-2.2361)^2 - 5}{2 \times (-2.2361)} = -2.2361 \quad (13)$$

So to four decimals, the other square root of 5 is  $-2.2361$ .

My dad notes that Eq. (5) for computing square roots was typically used in the equivalent form

$$x_{n+1} = \frac{1}{2} \left( x_n + \frac{k}{x_n} \right), \quad (14)$$

so that you repeatedly find the average of  $x_n$  and  $k/x_n$ .

## Further Reading

Numerical analysis is a whole subject in itself, and quite a number of books have been written about it. One good starting point is *Numerical Recipes* by Press, Teukolsky, Vetterling, and Flannery. It includes not only computer codes for various methods, but also a good discussion of the motivation behind the methods.