

The Factorial Function, $n!$

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Factorials

A function that occasionally appears in equations of mathematical physics is the *factorial function*, denoted by the notation $n!$, where n is a positive integer. It is defined as the product of all integers from 1 through (and including) n . Thus

$$\begin{aligned}1! &= 1 \\2! &= 1 \times 2 = 2 \\3! &= 1 \times 2 \times 3 = 6 \\4! &= 1 \times 2 \times 3 \times 4 = 24 \\5! &= 1 \times 2 \times 3 \times 4 \times 5 = 120 \\6! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720\end{aligned}$$

In product notation,

$$n! = \prod_{i=1}^n i \quad (1)$$

By convention, $0! = 1$.

Double Factorials

Somewhat less common is the *double factorial function*, $n!!$. This does occasionally occur, for example in a formula for the exact period of simple plane pendulum. The double factorial is defined as follows:

- If n is even, then $n!!$ is the product of all *even* integers from 2 to n .
- If n is odd, then $n!!$ is the product of all *odd* integers from 1 to n .

Thus

$$\begin{aligned}1!! &= 1 \\2!! &= 2 \\3!! &= 1 \times 3 = 3 \\4!! &= 2 \times 4 = 8 \\5!! &= 1 \times 3 \times 5 = 15 \\6!! &= 2 \times 4 \times 6 = 48\end{aligned}$$

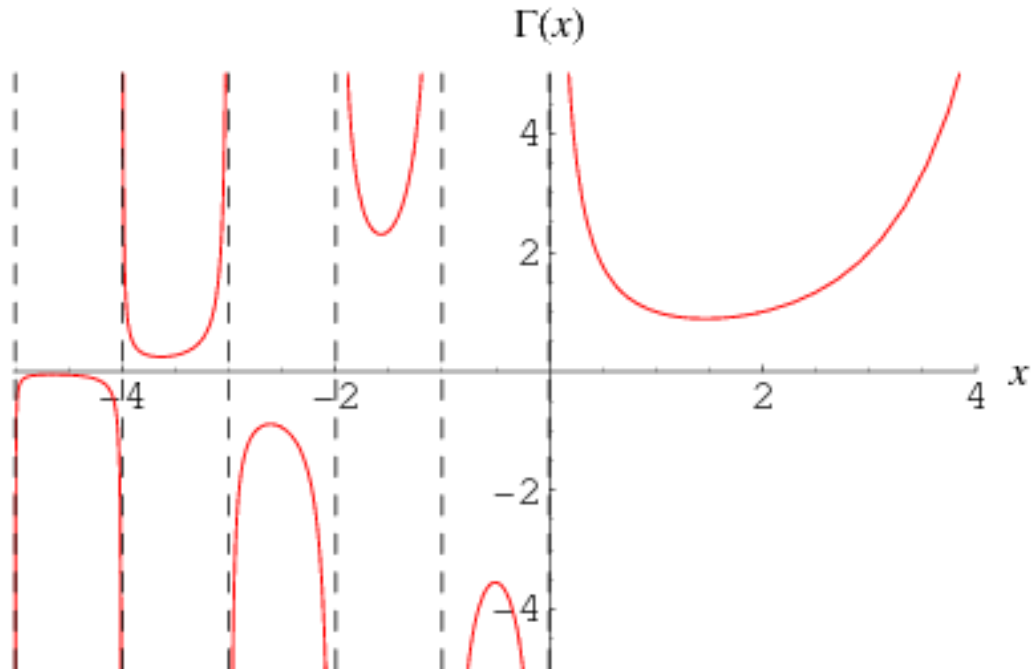


Figure 1: The gamma function, $\Gamma(x)$. (Credit: Wolfram MathWorld)

In product notation,

$$n!! = \begin{cases} \prod_{i=1}^{n/2} 2i & (n \text{ even}) \\ \prod_{i=1}^{(n+1)/2} (2i-1) & (n \text{ odd}) \end{cases} \quad (2)$$

By convention, $-1!! = 0!! = 1$.

The double factorial function can be related to the factorial function through the identities

$$(2n)!! = 2^n n! \quad (3)$$

$$(2n+1)!! = \frac{(2n+1)!}{2^n n!} \quad (4)$$

for any positive integer n . The first identity is for even double factorials, and the second is for odd double factorials.

The Gamma Function

It is possible to generalize the factorial function to include non-integers. The result is a special function called the *gamma function* $\Gamma(x)$, and is defined by

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt. \quad (5)$$

The gamma function is defined for all x except for 0 and negative integers.

A plot of the gamma function is shown in Figure 1.

The relation between the factorial function and the gamma function is

$$n! = \Gamma(n + 1), \quad (6)$$

where n is a non-negative integer. Thus $\Gamma(1) = 0! = 1$, $\Gamma(2) = 1! = 1$, $\Gamma(3) = 2! = 2$, $\Gamma(4) = 3! = 6$, etc.

One interesting relation involving the gamma function is

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \quad (7)$$

More generally, for integer $n \geq 0$,

$$\Gamma\left(\frac{1}{2} + n\right) = \frac{(2n - 1)!!}{2^n} \sqrt{\pi}, \quad (8)$$

so $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$, $\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}$, $\Gamma\left(\frac{7}{2}\right) = \frac{15}{8}\sqrt{\pi}$, etc.