The Factorial Function, $n!$

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Factorials

A function that occasionally appears in equations of mathematical physics is the *factorial function*, denoted by the notation $n!$, where n is a positive integer. It is defined as the product of all integers from 1 through (and including) n. Thus

$$\begin{aligned}
1! &= 1 \\
2! &= 1 \times 2 = 2 \\
3! &= 1 \times 2 \times 3 = 6 \\
4! &= 1 \times 2 \times 3 \times 4 = 24 \\
5! &= 1 \times 2 \times 3 \times 4 \times 5 = 120 \\
6! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720
\end{aligned}$$

In product notation,

$$n! = \prod_{i=1}^{n} i \tag{1}$$

By convention, $0! = 1$.

Double Factorials

Somewhat less common is the *double factorial function*, $n!!$. This does occasionally occur, for example in a formula for the exact period of simple plane pendulum. The double factorial is defined as follows:

- If n is even, then $n!!$ is the product of all *even* integers from 2 to n.
- If n is odd, then $n!!$ is the product of all *odd* integers from 1 to n.

Thus

$$\begin{aligned}
1!! &= 1 \\
2!! &= 2 \\
3!! &= 1 \times 3 = 3 \\
4!! &= 2 \times 4 = 8 \\
5!! &= 1 \times 3 \times 5 = 15 \\
6!! &= 2 \times 4 \times 6 = 48
\end{aligned}$$

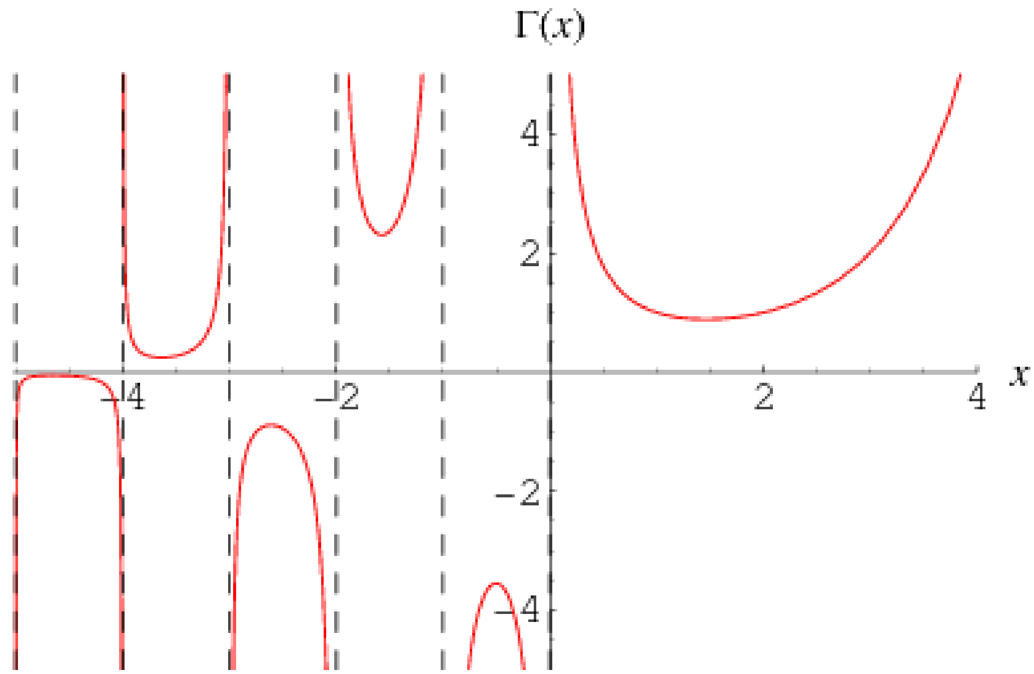

Figure 1: The gamma function, $\Gamma(x)$. *(Credit: Wolfram MathWorld)*

In product notation,

$$n!! = \begin{cases} \prod_{i=1}^{n/2} 2i & (n \text{ even}) \\ \prod_{i=1}^{(n+1)/2} (2i-1) & (n \text{ odd}) \end{cases} \tag{2}$$

By convention, $-1!! = 0!! = 1$.
The double factorial function can be related to the factorial function through the identities

$$(2n)!! = 2^n n! \tag{3}$$

$$(2n+1)!! = \frac{(2n+1)!}{2^n n!} \tag{4}$$

for any positive integer n. The first identity is for even double factorials, and the second is for odd double factorials.

The Gamma Function

It is possible to generalize the factorial function to include non-integers. The result is a special function called the *gamma function* $\Gamma(x)$, and is defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt. \tag{5}$$

The gamma function is defined for all x except for 0 and negative integers.
A plot of the gamma function is shown in Figure 1.

The relation between the factorial function and the gamma function is

$$n! = \Gamma(n+1), \tag{6}$$

where n is a non-negative integer. Thus $\Gamma(1) = 0! = 1$, $\Gamma(2) = 1! = 1$, $\Gamma(3) = 2! = 2$, $\Gamma(4) = 3! = 6$, etc.

One interesting relation involving the gamma function is

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}. \tag{7}$$

More generally, for integer $n \geq 0$,

$$\Gamma\left(\tfrac{1}{2} + n\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}, \tag{8}$$

so $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$, $\Gamma(\frac{5}{2}) = \frac{3}{4}\sqrt{\pi}$, $\Gamma(\frac{7}{2}) = \frac{15}{8}\sqrt{\pi}$, etc.