Dimensional Analysis

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Introduction

One of the most common types of errors made by both students and professional physicists while solving physics problems is to make errors in units of measure. By meticulously keeping track of the units you're working with, you can avoid quite a few problems.

We'll be working primarily with the SI system of units (so called because of its French name, *Système International d'Unités*). In the SI system, there are seven base units and two supplementary units (Table 1), with which all other units may be defined. Table 2 shows derived SI units with special names, along with their definitions. Each of these units may be modified with the prefixes shown in Table 3 to create units of convenient size. You should memorize the SI prefixes for tera- (10^{12}) through femto- (10^{-15}). The other prefixes are not used often.

One easy way to help avoid problems with units is to begin a problem by immediately converting all given quantities to base SI units—that is, units without prefixes except for kilograms. In other words, at the beginning of a problem, immediately convert all lengths to meters, all times to seconds, all masses to kilograms, all energies to joules, all forces to newtons, etc. Then your final answer will be in base units also. (For example, if the final answer to the problem is a velocity, then it will automatically come out in m/s.)

Rules of Dimensional Analysis

Units can be used as a kind of "debugging" technique in working problems. For example, when deriving an equation, you should check the units on your final answer; if the units are correct, you *may* have the correct answer, but if the answers are wrong, you *must* have the *wrong* answer. If the final answer has the wrong units, you can examine the units of the intermediate steps; the step where the units start to go wrong is the place where you made a mistake.

There are a few rules to keep in mind when working with units:

1. The units must match on both sides of an equation. (To check this, it may sometimes be helpful to break everything down to base units, with the help of Table 2.)

2. Units of radians or steradians may be considered dimensionless.

3. Whenever you add or subtract two quantities, the units must match. (It's OK to multiply or divide different units.)

4. The arguments of some functions must be dimensionless (or be in radians). For example, x must be dimensionless in $\sin x$, $\cos x$, $\tan x$, $\log x$, and a^x.

Examples

Let's look at a few examples of the sorts of situations that may arise in dimensional analysis:

1. Suppose we derive an equation in mechanics that says $v = at^2$. The left-hand side has units of m/s, and the right-hand side has units of (m/s^2)(s^2)= m. Since the units on each side of the equation are different, there must be an error.

2. Consider the equation in rotational mechanics for angular momentum: $L = I\omega$. On the left-hand side, angular momentum L has units of N·m·s. On the right-hand side, I has units of kg·m^2 and ω has units of rad/s, so the right-hand side has units of kg·m^2·rad/s, or N·m·s·rad. The units on both sides match except for an extra unit of radians on the right-hand side, which is OK—radians can be considered dimensionless.

3. Consider the equation $E = W + mv$. On the right-hand side, work W in joules is being added to mv, which has units of kg·m/s. Since you can't add two quantities whose units don't match, there is an error on the right-hand side.

4. Consider the equation $y = \sin(\omega t^2)$. The argument of the sine function has units of rad·s, so there is an error here, since the argument of the sine function must be dimensionless (or be in radians).

5. Consider the equation $C = \varepsilon_0 A/d$, where C is in farads, ε_0 is in C^2 N^{-1} m^{-2}, A is in m^2, and d is in m. Do the units match on both sides of the equation? Try breaking everything down into base units. On the left-hand side, we use Table 2 to find farads are A^2 s^4 kg^{-1} m^{-2}. On the right-hand side, we have (A^2 s^2)(s^2/(kg m))(1/m^2)(m^2)(1/m), which is A^2 s^4 kg^{-1} m^{-2}, so the units on both sides do indeed match.

SI Units Tables

Table 1. SI base units and supplementary units.

Name	Symbol	Quantity
meter	m	length
kilogram	kg	mass
second	s	time
ampere	A	electric current
kelvin	K	temperature
mole	mol	amount of substance
candela	cd	luminous intensity
radian	rad	plane angle
steradian	sr	solid angle

Table 2. Derived SI units.

Name	Symbol	Definition	Base Units	Quantity
newton	N	kg m s^{-2}	kg m s^{-2}	force
joule	J	N m	kg m^2 s^{-2}	energy
watt	W	J / s	kg m^2 s^{-3}	power
pascal	Pa	N / m^2	kg m^{-1} s^{-2}	pressure
hertz	Hz	s^{-1}	s^{-1}	frequency
coulomb	C	A s	A s	electric charge
volt	V	J / C	kg m^2 A^{-1} s^{-3}	electric potential
ohm	Ω	V / A	kg m^2 A^{-2} s^{-3}	electrical resistance
siemens	S	A / V	kg^{-1} m^{-2} A^2 s^3	electrical conductance
farad	F	C / V	kg^{-1} m^{-2} A^2 s^4	capacitance
weber	Wb	V s	kg m^2 A^{-1} s^{-2}	magnetic flux
tesla	T	Wb / m^2	kg A^{-1} s^{-2}	magnetic field strength
henry	H	Wb / A	kg m^2 A^{-2} s^{-2}	induction
lumen	lm	cd sr	cd sr	luminous flux
lux	lx	lm / m^2	cd sr m^{-2}	illuminance
becquerel	Bq	s^{-1}	s^{-1}	radioactivity
gray	Gy	J / kg	m^2 s^{-2}	absorbed dose
sievert	Sv	J / kg	m^2 s^{-2}	dose equivalent
katal	kat	mol / s	mol s^{-1}	catalytic activity

Table 3. SI prefixes.

Prefix	Symbol	Definition	English
yotta-	Y	10^{24}	septillion
zetta-	Z	10^{21}	sextillion
exa-	E	10^{18}	quintillion
peta-	P	10^{15}	quadrillion
tera-	T	10^{12}	trillion
giga-	G	10^{9}	billion
mega-	M	10^{6}	million
kilo-	k	10^{3}	thousand
hecto-	h	10^{2}	hundred
deka-	da	10^{1}	ten
deci-	d	10^{-1}	tenth
centi-	c	10^{-2}	hundredth
milli-	m	10^{-3}	thousandth
micro-	μ	10^{-6}	millionth
nano-	n	10^{-9}	billionth
pico-	p	10^{-12}	trillionth
femto-	f	10^{-15}	quadrillionth
atto-	a	10^{-18}	quintillionth
zepto-	z	10^{-21}	sextillionth
yocto-	y	10^{-24}	septillionth