Celestial Mechanics

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1 Introduction

The area of classical mechanics that deals with the orbits of astronomical bodies around each other under the influence of gravity is called *celestial mechanics*. Celestial mechanics is a vast (and very interesting) field; here we'll get just a taste for how to do some basic calculations, where we examine a simple orbit of one body around another — the so-called "two-body problem".

2 Time

The way we measure time for civil use (years, months, days, weeks, etc.) is not particularly convenient for astronomical calculations. A more convenient way to measure time is with the *Julian Day*. The Julian Day is simply a count of the total number of days that have elapsed since *noon* on January 1, 4713 B.C. (by the old Julian calendar). (Notice that the Julian Day begins at noon, not at midnight as on our civil calendar.) As an example, December 1, 2010 (midnight, beginning of December 1) is Julian Day 2455531.5.

The calendar date may be converted to and from the Julian Day using some fairly simple, well-known algorithms (see e.g. Meeus, 1991), or by the use of pre-computed tables.

The Julian Day makes it very easy to find the number of days between two dates: just convert both dates to their corresponding Julian Day, and subtract. This is how computer programs like spreadsheets deal with dates: they store dates internally as Julian Days, and use standard algorithms to convert to and from the calendar date that is displayed on the screen.

3 Orbit Reference Frames

In order to describe the orientation of an orbit in space, we need to have a reference frame with respect to which the orbit will be described. Such a reference system is defined by a reference *plane*, and a reference *direction* that lies in that plane. The two common choices are the *equatorial* and *ecliptic* reference frames.

In the *equatorial* reference frame, the reference plane is the plane of the Earth's equator, and the reference direction is the direction of the *vernal equinox*. The vernal equinox is the direction from the Earth to the Sun on the first day of spring (around March 21). This equatorial frame is commonly used for bodies orbiting the Earth, such as artificial satellites.

In the *ecliptic* reference frame, the reference plane is the plane of the *ecliptic* (i.e. the plane of the Earth's orbit around the Sun), and the reference direction is again in the direction of the vernal equinox. The ecliptic frame is used for most astronomical bodies: planets, comets, etc.

The plane of the equator and the plane of the ecliptic intersect along a line, and the direction of the vernal equinox lies along that line of intersection. The two planes are separated by a dihedral angle of about 23.5° (the tilt angle of the Earth's axis); this angle is called the *obliquity of the ecliptic* (ε).

4 Orbital Elements

Now suppose that we want to describe the orbit of one body around another: for example, the Moon around the Earth, or the planet Saturn around the Sun. We first choose an appropriate reference frame, and then we need to describe the orbit. The orbit is specified using a set of seven numbers called the *orbital elements* of the orbit, which are described here.

Figure 1 shows a typical orbit and reference frame. In this figure, the xy-plane is the reference frame (either the equator or the ecliptic), and the x direction is the reference direction (the vernal equinox). The orbit plane intersects the reference plane along a line called the *line of nodes*. The point where the orbiting body moves from below the reference plane to above the reference plane is called the *ascending node*, and is marked N in Fig. 1. The opposite point on the line of nodes, where the body moves from above the reference plane to below is called the *descending node*.

The point of closest approach of the orbiting body to the center body is called the *pericenter*, and the point of farthest approach is called the *apocenter*. In the case where the body is orbiting the Earth, these are called the *perigee* and *apogee* (respectively); when the body is orbiting the Sun, these points are called the *perihelion* and *aphelion* (respectively). The line connecting the pericenter to the apocenter is called the *line of apsides*.

Now to the orbital elements. First, we need to specify the *size* of the orbit. Bodies in closed orbits always orbit in *ellipses*, where the body being orbited is at one of the two foci of the ellipse. The size of the orbit is specified by giving the *semi-major axis* a of the ellipse.

Second, we need to specify the *shape* of the orbit. We do this by specifying the *eccentricity* e of the ellipse. The eccentricity is a number between 0 and 1, and is measure of how elongated the ellipse is: $e = 0$ for a circle, and values of e close to 1 are long, cigar-shaped ellipses. The eccentricity e is related to the semi-major axis a and semi-minor axis b of the ellipse by

$$e = \frac{\sqrt{a^2 - b^2}}{a}. \tag{1}$$

Next, we need to specify the *orientation* of the orbit in space. This requires three angles: (1) the *inclination* i of the orbit with respect to the reference plane; (2) the *longitude of the ascending node* Ω, which is the angle between the vernal equinox and the ascending node, measured in the reference plane; and (3) the *argument of pericenter* ω, which is the angle between the ascending node and the orbit pericenter, measured in the plane of the orbit. These three angles are illustrated in Fig. 1.

Now we've completely specified the orbit itself, but we need one more bit of information: *where* the body is in this orbit. This requires two numbers: an angle, and a time at which the body is at that angle. The angle is called the *mean anomaly at epoch* M_0, and gives the angle from the pericenter to the body (measured in the plane of the orbit) at a specified *epoch time* T_0.

The seven orbital elements are summarized in the table below, and illustrated in Figure 1.

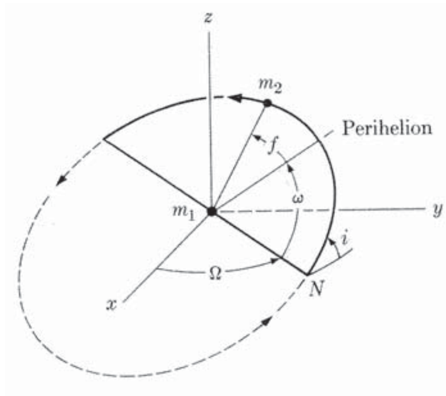

Figure 1: Orbital elements for a body of mass m_2 orbiting a body of mass m_1. The xy-plane is the reference plane, and x is the direction of the vernal equinox. Shown are the orbital elements i, Ω, and ω, along with the true anomaly f. Point N is the ascending node of the orbit. (From McCuskey, 1963.)

Table 1. Orbital elements.

Element	Symbol
Semi-major axis	a
Eccentricity	e
Inclination	i
Longitude of ascending node	Ω
Argument of pericenter	ω
Mean anomaly at epoch	M_0
Epoch time	T_0

5 Right Ascension and Declination

The goal of the orbit calculation is to find the position of a body in the sky, given its orbital elements. The final result, the position in the sky, will be given in a coordinate system that is analogous to the longitude-latitude system used to locate places on the surface of the Earth. Imagine rotating the Earth on its axis until the prime meridian (0° longitude) intersects the direction of the vernal equinox. Then projecting the lines of geographic longitude into the sky gives lines of *right ascension* for astronomical objects. Similarly, projecting the lines of geographic latitude into the sky give lines of *declination*.

Here's another way to think of this: imagine the Earth is a hollow glass sphere, with longitude and latitude lines drawn on it. Rotate the Earth on its axis until the prime meridian intersects the direction of the vernal equinox, and hold the Earth still at that position. Now if you are at the center of the Earth and look out toward the sky, the lines drawn on the glass will be lines of right ascension and declination.

Right ascension ranges from 0° to 360°, and declination ranges from $-90°$ to $+90°$ (where $+$ is north). Often right ascension is given in units of *hours*, rather than degrees (1 hour $= 15°$). Under this convention, right ascension ranges from 0h to 24h.

6 Computing a Position

Now let's put all this together and see how you go about computing the position of a body — let's say a planet orbiting the Sun — at a time t, given its orbital elements. We begin by computing the *mean daily motion* n of the body, which is how many revolutions it makes in its orbit per day. This is found from Kepler's Third Law:

$$n = \frac{86400}{2\pi}\sqrt{\frac{GM_\odot}{a^3}}, \tag{2}$$

where G is the universal gravitational constant, $M_\odot$ is the mass of the central body, and a is the semi-major axis of the orbit. The factor $86400/2\pi$ in front converts to units of rev/day.

Next we find the *mean anomaly* M at time t:

$$M = M_0 + 2\pi n(t - T_0) \tag{3}$$

Essentially what we're doing here is assuming the orbit is a perfect circle; knowing the mean anomaly M_0 at the epoch time T_0, this equation finds the mean anomaly M at some other time t. Here both M and M_0 are in units of radians, t and T_0 are Julian Days, and n is in units of rev/day.

Of course, the real orbit is generally an ellipse, not a circle, so the next step is to adjust the mean anomaly M to correct it for the eccentricity of the orbit. The result is called the *eccentric anomaly* E. We find the eccentric anomaly by solving the following equation, called *Kepler's equation*, for E:

$$M = E - e\sin E \tag{4}$$

(Here M and E are both in radians.) Kepler's equation cannot be solved for E in closed form, so we need to make use of some iterative method such as Newton's method to solve for E.

Having found E, the next step is to correct the orbit for the fact that the body is at one of the foci of the ellipse, not at the center of the ellipse. This correction gives what's called the *true anomaly* f (again in radians):

$$\tan\left(\frac{f}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E}{2}\right) \tag{5}$$

The true anomaly f is the true polar coordinate of the body, measured from the pericenter to the body, in the plane of the orbit.

Next we find radial distance r of the orbiting body from the central body:

$$r = a(1 - e\cos E) \tag{6}$$

The quantities r and f are the plane polar coordinates of the orbiting body, with the central body at the origin. The remainder of the calculations are essentially a set of coordinate transformations to find the right ascension and declination of the body.

We begin these coordinate transformations by finding the *argument of latitude* u (radians):

$$u = \omega + f \tag{7}$$

Next, we find the heliocentric cartesian ecliptic coordinates (x, y, z) of the orbiting body:

$$x = r(\cos u\cos\Omega - \sin u\sin\Omega\cos i) \tag{8}$$

$$y = r(\cos u\sin\Omega + \sin u\cos\Omega\cos i) \tag{9}$$

$$z = r\sin u\sin i \tag{10}$$

For the orbit of a planet around the Sun, these are the cartesian coordinates of the body in a coordinate system centered at the Sun.

We don't want to know where the body will appear in the sky as seen from the Sun, though—we want to know where it will be in the sky as seen from the Earth. So next we move the origin of this coordinate system from the Sun to the Earth, giving the geocentric cartesian ecliptic coordinates (x_e, y_e, z_e) of the body:

$$x_e = x + x_\odot \tag{11}$$

$$y_e = y + y_\odot \tag{12}$$

$$z_e = z + z_\odot \tag{13}$$

where $(x_\odot, y_\odot, z_\odot)$ are the geocentric cartesian coordinates of the Sun at time t.

Now we convert from cartesian to spherical coordinates. Assuming the reference plane is the elciptic, this gives the geocentric *ecliptic longitude* λ and *ecliptic latitude* β:

$$\tan\lambda = \frac{y_e}{x_e} \tag{14}$$

$$\sin\beta = \frac{z_e}{\sqrt{x_e^2 + y_e^2 + z_e^2}} \tag{15}$$

Finally, we convert these ecliptic coordinates to right ascension α and declination δ:

$$\tan\alpha = \frac{\sin\lambda\cos\varepsilon - \tan\beta\sin\varepsilon}{\cos\lambda} \tag{16}$$

$$\sin\delta = \sin\beta\cos\varepsilon + \cos\beta\sin\varepsilon\sin\lambda, \tag{17}$$

where ε is the obliquity of the ecliptic (about 23.5°).

Eqs. (16) and (17) are the solution to the problem: we could find a star atlas (which has lines of right ascension and declination marked on it), locate the planet, and find where in the sky the planet can be seen.

7 The Inverse Problem

The problem we just solved is: given the orbital elements of the planet, we found its position in the sky at any given time. But how did we get the orbital elements in the first place? This has to do with the inverse of the problem just solved: given the position of the planet in the sky, what are the orbital elements?

It turns out that we require *three* separate observations of the body at three different times. Knowing the right ascension α and declination δ of the body at three different times, one can derive the orbital elements. Details are given in chapter 4 of the reference by McCuskey.

8 Further Topics

We've described here the basics of a two-body orbit calculation, but there are a number of corrections that would need to be made to make a more accurate calculation; for example:

- The reference frames are actually not fixed, but move in time because of motions of the Earth. A more careful calculation would take these effects (precession and nutation of the Earth) into account.
- The orbital elements change with time — notably the longitude of the ascending node and the argument of pericenter.
- Other bodies are always present – not just the planet and the Sun. More complex calculations take the effect of other bodies into account.

- Parallax corrections: the position of the body in the sky varies slightly depending on the position of the observer on the surface of the Earth.

- Atmospheric refraction can cause small changes in the apparent position of the body in the sky.

9 References

- *Introduction to Celestial Mechanics* by S.W. McCuskey. Addison-Wesley, Reading, Mass., 1963. A brief, excellent introduction to celestial mechanics.

- *Astronomical Algorithms* by Jean Meeus. Willmann-Bell, Richmond, 1991. Another excellent book, with 58 chapters of material covering how to do practical calculations of all sorts related to celestial mechanics.

- *The Astronomical Almanac*. U.S. Government Printing Office. This is published in a new edition each year, and is full of data related to celestial mechanics.