The Calculus

D.G. Simpson, Ph.D.
Department of Physical Sciences and Engineering
Prince George's Community College

August 1, 2019

Some ideas in physics are most naturally expressed in terms of a branch of mathematics called *the calculus of infinitesimals*, or simply *the calculus*. Here we will present a very brief overview of the ideas of the calculus so that the notation will be familiar when we encounter it. For a more complete, rigorous, and in-depth understanding of the calculus, the student is referred to courses on the subject.

1 Infinitesimal Numbers

Briefly stated, *the calculus is the mathematics of infinitesimal numbers.* Infinitesimal numbers are an extension to the set of real numbers. Following Leibniz, we will call an infinitesimal number on the number line (the x axis) by the notation dx. The symbol dx is to be thought of a one symbol; it does *not* mean $d \times x$.

Here's another way to think of the infinitesimal number dx. You've probably encountered the "Δ" notation before, meaning the difference between two real numbers. For example, if $x_1 = 3$ and $x_2 = 7$, then $\Delta x = x_2 - x_1 = 7 - 3 = 4$ is their difference. The notation dx is analogous to Δx, but refers to the difference between two numbers that are "infinitely close together."

Mathematically, we define the infinitesimal number dx by

$$\exists\, dx \;:\; 0 < dx < x, \forall x \in \mathbb{R} \tag{1}$$

In other words, *the (positive) infinitesimal number dx is greater than zero, but smaller than any real number*. You may wonder how this is possible. The answer is: it's just defined this way. Mathematicians have determined that infinitesimal numbers can be defined this way without mathematical contradiction.

Intuitively, you can think of the infinitesimal number dx as being "infinitely close" to zero, but *not* zero. Think of dx as a *very, very, very, very* small number — an "infinitely small" number.

Infinitesimal numbers obey many of the expected laws of arithmetic. Addition and subtraction work as you would expect:

$$dx + dx = 2dx \tag{2}$$

$$2dx + dx = 3dx \tag{3}$$

$$3dx - dx = 2dx \tag{4}$$

Multiplication is also defined:

$$dx \times dx = (dx)^2 \tag{5}$$

The number $(dx)^2$ is also an infinitesimal number, but is "infinitely smaller" than dx. This is as expected: if we approximate dx by a very small number like 10^{-6}, then its square (10^{-12}) is much smaller in comparison.

Division of infinitesimals leads to some interesting results. In general, dividing one infinitesimal number by another often leads to a *finite* result, as we'll see in the next section.

2 Differential Calculus — Finding Slopes

One important application of the calculus is that it allows us to determine the slope of a line that is not necessarily a straight line. You've learned in an algebra class how to find the slope of a straight line:

$$\text{slope} = \frac{\text{rise}}{\text{run}} \tag{6}$$

In other words, pick any two points along the line, and take the change in y (Δy, the "rise") divided by the change in x (Δx, the "run").

How can you calculate the slope of a line that is *not* straight — say, for example, the parabola $y = x^2$? For a curved line, the slope is different at different points along the curve; it is defined to be the slope of the straight line tangent to the curve at that point. We can calculate the slope of that tangent line by using the calculus.

As an example, let's take the parabola $f(x) = x^2$ and say we wish to find its slope at $x = 3$. We can approximate the slope of the tangent line at $x = 3$ by finding the slope of the straight line connecting the point on the parabola at $x = 3$ and a second point very close to $x = 3$. The closer the second point is to $x = 3$, the better the approximation to the actual slope *at* $x = 3$. For example, let the two points be $x = 3$ and $x = 3.01$. Then at $x = 3$, $y = f(x) = x^2 = 3^2 = 9$, and at $x = 3.01$, $y = f(x) = x^2 = 3.01^2 = 9.0601$. The slope of the line connecting these points is then

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{9.0601 - 9}{3.01 - 3} = 6.01 \tag{7}$$

Now let's try an even closer second point: $x = 3.001$. Then $y = x^2 = 3.001^2 = 9.006001$. Then

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{9.006001 - 9}{3.001 - 3} = 6.001 \tag{8}$$

And yet an even closer second point: $x = 3.0001$. Then $y = x^2 = 3.0001^2 = 9.00060001$. Then

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{9.00060001 - 9}{3.0001 - 3} = 6.0001 \tag{9}$$

The closer the second point is to 3, the closer the slope seems to be getting to 6. In other words, in the *limit* where Δx gets closer and closer to 0, the slope gets closer and closer to 6 — suggesting that the slope *at* $x = 3$ is *exactly* 6. We write this limit as:

$$\text{slope} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{(x + \Delta x) - x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{10}$$

Since $f(x) = x^2$ in our example,

$$\text{slope} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{11}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} \tag{12}$$

$$= \lim_{\Delta x \to 0} \frac{[x^2 + 2x\Delta x + (\Delta x)^2] - x^2}{\Delta x} \tag{13}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \tag{14}$$

Canceling Δx in the numerator and denominator,

$$\text{slope} = \lim_{\Delta x \to 0} 2x + \Delta x \tag{15}$$

and as Δx approaches zero,

$$\text{slope} = 2x \tag{16}$$

So for at any point along the curve $f(x) = x^2$, its slope is given by $2x$. At $x = 3$, the slope is $2 \times 3 = 6$, in agreement with our earlier approximations.

The slope is called the *derivative* of $f(x)$ with respect to x. As we have just shown, the derivative of $f(x) = x^2$ with respect to x is $2x$. We indicate the derivative of $y = f(x)$ with respect to x by the notation

$$\frac{dy}{dx} \text{ or } \frac{d}{dx}\, f(x) \tag{17}$$

Thus the derivative can be thought of as the quotient of two infinitesimal numbers, and is defined as

$$\frac{dy}{dx} \equiv \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{18}$$

For our example $y = f(x) = x^2$,

$$\frac{dy}{dx} = \frac{d}{dx}\, x^2 = 2x \tag{19}$$

More generally, it can be shown that for any n,

$$\frac{d}{dx}\, x^n = n x^{n-1} \tag{20}$$

For example,

$$\frac{d}{dx}\, x^5 = 5x^4 \tag{21}$$

Here n need not necessarily be an integer. For example, since $\sqrt{x} = x^{1/2}$, we have

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}\, x^{1/2} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}} \tag{22}$$

Similar results can be worked out for many common functions. Section 9 gives a short table of derivatives. In conjunction with this table, we note the following properties (u and v are functions of x, and a is a constant):

$$\frac{d}{dx}\,(au) = a\,\frac{du}{dx} \tag{23}$$

$$\frac{d}{dx}\,(u + v) = \frac{du}{dx} + \frac{dv}{dx} \tag{24}$$

$$\frac{d}{dx}\,(u - v) = \frac{du}{dx} - \frac{dv}{dx} \tag{25}$$

$$\frac{d}{dx}\,(uv) = \frac{du}{dx}\, v + u\,\frac{dv}{dx} \tag{26}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v(du/dx) - u(dv/dx)}{v^2} \tag{27}$$

These results will be proved in a more rigorous calculus course.

Now we know how to find the slope of a line that is non necessarily straight: find a formula for the derivative of the curve, and the slope at any point is the derivative evaluated at that point. Why would we want to find the slope of a curved line? For one thing, a derivative with respect to time is how we describe the rate of change of something. For example, velocity is the rate of change of position, so the velocity of a body is written in terms of the derivative of its position with respect to time: $v = dx/dt$ — so that if you have a function $x(t)$ that gives the position x of a body at any time t, you can take the derivative with respect to t and get a formula that gives the velocity v of the body at any time t. Another use for the derivative is for optimization problems: the tangent at the peak of a curve is equal to zero, so to locate the peak of a curve, we calculate its derivative and set it equal to zero.

Here's an interesting calculus fact: there's one function that is equal to its own derivative. That function is e^x:

$$\frac{d}{dx} e^x = e^x \tag{28}$$

Example. Find the derivative of the function $f(x) = 4x^3 + 7x^2 - 5x + 6$ with respect to x, and find the slope of $f(x)$ at $x = 3$.

Solution. Using the above results,

$$\begin{aligned}
\frac{d}{dx} f(x) &= \frac{d}{dx} (4x^3 + 7x^2 - 5x + 6) &(29)\\
&= \frac{d}{dx}(4x^3) + \frac{d}{dx}(7x^2) - \frac{d}{dx}(5x) + \frac{d}{dx}(6) &(30)\\
&= 4\frac{d}{dx}(x^3) + 7\frac{d}{dx}(x^2) - 5\frac{d}{dx}(x) + \frac{d}{dx}(6) &(31)\\
&= 4(3x^2) + 7(2x) - 5 + 0 &(32)\\
&= 12x^2 + 14x - 5 &(33)
\end{aligned}$$

The slope at $x = 3$ is then $12(3)^2 + 14(3) - 5 = 145$.

Example. Locate the peaks of the function $f(x) = 4x^3 + 7x^2 - 5x + 6$.

Solution. The peaks are where the derivative is equal to zero. We found the derivative in the previous example, so set this derivative equal to zero to find the peaks:

$$12x^2 + 14x - 5 = 0 \tag{34}$$

By the quadratic formula,

$$x = \frac{-14 \pm \sqrt{14^2 - 4 \times 12 \times (-5)}}{2 \times 12} = \frac{-7 \pm \sqrt{109}}{12} = \{-1.4534,\ 0.2867\} \tag{35}$$

This gives the two values of x at which the peaks are located.

3 Integral Calculus — Finding Areas

Besides finding slopes, another application of the calculus is the find the *area* under a curve (i.e. between the curve and the x axis). The area under a *straight* line is easy to find without the calculus: it's just the area of a trapezoid. But under a *curved* line, we use the calculus to compute the area.

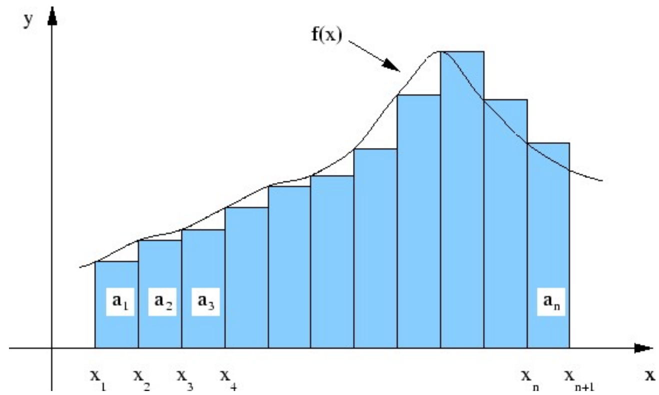

Figure 1: Finding the area under a curve using rectangles (*Credit: pleacher.com*)

To do this, imagine dividing the area under the curve into a number of very thin rectangles (Figure 1). The thinner the rectangles, the more rectangles we have, and the better the approximation to the actual area under the curve.

If we go to the limit where the rectangles are infinitesimally narrow, then we will have infinitely many of them, and the sum of the areas of all the rectangles exactly equals the area under the curve. Adding up an infinite number of infinitesimal numbers is called *integration*, and typically results in a finite result. If we have a curve $f(x)$, then a rectangle at x has infinitesimal width dx and finite height $f(x)$, so that that rectangle has area equal to its width times its height, or $f(x)\,dx$. We add together an infinite number of them by integration; the symbol for which is an elongated S (for "sum"), $\int$:

$$\int f(x)\,dx \tag{36}$$

This expression is called an *integral*, and the function $f(x)$ is called the *integrand* of the integral. The area under the curve clearly depends on where the left and right ends of the area are. The area under the curve $f(x)$ between $x = a$ and $x = b$ in indicated by

$$\int_a^b f(x)\,dx \tag{37}$$

Equation (36) is called an *indefinite integral*, and Equation (37) is called a *definite integral*. To compute a definite integral, we evaluate the *indefinite* integral at the upper bound b, and subtract the indefinite integral evaluated at the lower bound a:

$$\int_a^b f(x)\,dx = \int f(x)\,dx \text{ (at } x = b) \;-\; \int f(x)\,dx \text{ (at } x = a) \tag{38}$$

For example, suppose we want to find the area under the parabola $f(x) = x^2$ between $x = 1$ and $x = 3$. This would be

$$\text{area} = \int_1^3 x^2\,dx = \left(\frac{x^3}{3}\right)\Bigg|_1^3 = \frac{3^3}{3} - \frac{1^3}{3} = \frac{26}{3} \text{ square units} \tag{39}$$

The vertical bar is used to indicate that we evaluate the expression at the top value (3), then subtract the expression evaluated at the bottom value (1).

It is important to note that the area under the curve counts as *negative* area if it lies below the x axis. For example, consider a sine curve, $f(x) = \sin x$. The function $\sin x$ has a positive "lobe" above the x axis from $x = 0$ to $x = \pi$, and a negative "lobe" beneath the x axis from $x = \pi$ to $x = 2\pi$. If we find the integral of $f(x) = \sin x$ from $x = 0$ to $x = 2\pi$, we're finding the total area under the curve, but counting the part below the x axis as *negative*. We get (using Section 10):

$$\int_0^{2\pi} \sin x \, dx = (-\cos x)\Big|_0^{2\pi} = -\cos 2\pi - (-\cos 0) = -1 - (-1) = 0. \tag{40}$$

so the positive area of the first lobe is exactly cancelled by the negative area of the second lobe, and the total area under the curve is zero. If we really wanted to find the total area under the sine curve from $x = 0$ to $x = 2\pi$, counting all area as positive, we could find the area under just one positive lobe and double it:

$$\text{area} = 2\int_0^{\pi} \sin x \, dx = 2(-\cos x)\Big|_0^{\pi} = 2[(-\cos \pi) - (-\cos 0)] = 2[1 - (-1)] = 2 \times 2 = 4 \text{ sq. units} \tag{41}$$

The area under each lobe is 2 square units.

An unexpected result from the calculus is that the derivative (slope) and integration (area) are *inverse* operations of each other:

$$\frac{d}{dx} \int f(x) \, dx = f(x) \tag{42}$$

so the integral can be thought of as the "anti-derivative." This result is called the *fundamental theorem of calculus*.

In a rigorous calculus course, you will learn how to work out formulas for a number of simple functions. For example,

$$\int x^2 \, dx = \frac{x^3}{3} + C \tag{43}$$

where C is an arbitrary constant. All *indefinite* integrals will include this arbitrary constant, because when we take the inverse (a derivative), the derivative of this constant is zero. In effect, some information about the original function is lost when computing its derivative, so that you can't entirely recover the original function when computing the integral of the derivative. This lost information is expressed as an arbitrary constant C added to the indefinite integral. To find what C is, we would need some additional information, such as what value the integral is supposed to have at a specific point.

More generally,

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \tag{44}$$

As with the similar formula for derivatives, n need not be an integer. For example, since $\sqrt{x} = x^{1/2}$, we have

$$\int \sqrt{x} \, dx = \int x^{1/2} \, dx = \frac{x^{3/2}}{3/2} + C = \frac{2}{3}\sqrt{x^3} + C \tag{45}$$

Similar results can be worked out for many common functions. Section 10 gives a short table of integrals. In conjunction with this table, we note the following properties (u and v are functions of x, and a is a constant):

$$\int au \, dx = a \int u \, dx \tag{46}$$

$$\int (u + v) \, dx = \int u \, dx + \int v \, dx \tag{47}$$

$$\int (u - v) \, dx = \int u \, dx - \int v \, dx \tag{48}$$

These results will be proved in a more rigorous calculus course. There are no product or quotient rules for integrals as there are for derivatives.

Since the derivative and integration are inverses of each other, and the function e^x is equal to its own derivative, it is also equal to its own integral (to within an arbitrary constant of integration):

$$\int e^x\,dx = e^x + C \tag{49}$$

Example. Find the indefinite integral of the function $f(x) = 4x^3 + 7x^2 - 5x + 6$ with respect to x, and find the area under $f(x)$ between $x = 3$ and $x = 4$.

Solution. Using the above results,

$$\int f(x)\,dx = \int (4x^3 + 7x^2 - 5x + 6)\,dx \tag{50}$$

$$= \int 4x^3\,dx + \int 7x^2\,dx - \int 5x\,dx + \int 6\,dx \tag{51}$$

$$= 4\int x^3\,dx + 7\int x^2\,dx - 5\int x\,dx + 6\int dx \tag{52}$$

$$= 4\left(\frac{x^4}{4}\right) + C_1 + 7\left(\frac{x^3}{3}\right) + C_2 - 5\left(\frac{x^2}{2}\right) + C_3 + 6(x) + C_4 \tag{53}$$

$$= x^4 + \frac{7}{3}x^3 - \frac{5}{2}x^2 + 6x + C \tag{54}$$

where we have combined all the individual constants of integration C_1, C_2, C_3, C_4 into a single constant C.

To find the area under the curve between $x = 3$ and $x = 4$, we compute the definite integral

$$\int_3^4 f(x)\,dx \tag{55}$$

We've already found the indefinite integral; all we need to do is evaluate the indefinite integral at $x = 4$, and subtract the indefinite integral evaluated at $x = 3$:

$$\text{area} = \int_3^4 f(x)\,dx = \left(x^4 + \frac{7}{3}x^3 - \frac{5}{2}x^2 + 6x + C\right)\Bigg|_3^4 \tag{56}$$

$$= \left[(4)^4 + \frac{7}{3}(4)^3 - \frac{5}{2}(4)^2 + 6(4) + C\right] - \left[(3)^4 + \frac{7}{3}(3)^3 - \frac{5}{2}(3)^2 + 6(3) + C\right] \tag{57}$$

$$= \frac{1499}{6} \tag{58}$$

Notice that the constant of integration C always cancels out in a definite integral.

4 The Fundamental Theorem of Calculus

The *fundamental theorem of calculus* states an unexpected result: the derivative (slope-finding) and integral (area-finding) are inverses of each other. Thus

$$\frac{d}{dx}\int f(x)\,dx = f(x) \tag{59}$$

5 Approximations

It may sometimes happen that we have *data points* for which we need to calculate a derivative or integral. For example, suppose we have the following data for a moving body:

Time t (s)	Position x (m)
0.0	0.0
1.0	0.34
2.0	1.36
3.0	3.06
4.0	5.44
5.0	8.50
6.0	12.24

What is the velocity v of the body at time $t = 2.5$ seconds? By definition, the velocity v is found by a derivative: $v = dx/dt$. One way to *approximate* this derivative is by finding $\Delta x/\Delta t$, for the interval from 2.0 to 3.0 seconds,

$$\frac{dx}{dt} \approx \frac{\Delta x}{\Delta t} = \frac{3.06\text{ m} - 1.36\text{ m}}{3.0\text{ s} - 2.0\text{ s}} = 1.70\text{ m/s} \tag{60}$$

We could do the same for every time interval in the table, and use the midpoint of the time intervals as the time. We get the following table:

Time t (s)	Velocity v (m/s)
0.5	0.34
1.5	1.02
2.5	1.70
3.5	2.38
4.5	3.06
5.5	3.74

If the data in the table is "noisy" (has lots of measurement errors), then this kind of computing derivatives numerically can lead to very noisy results: small measurement errors can lead to a large change in slope from one point to the next.

Integrals can be computed numerically as well. There are a number of methods for doing this; the simplest is called the *rectangular rule*, in which we imaging drawing a rectangle at each data point, and approximate the integral as the sum of the rectangle areas. For example, for the body we've been using for our example, how far does the body travel from time $t = 0$ to time $t = 6$ seconds? That can be found as an integral:

$$x = \int_0^6 v(t)\,dt \approx \sum_{t=0}^{6} v(t)\,\Delta t \tag{61}$$

Using the data from the above table of velocities,

$$x \approx \sum_{t=0}^{6} v(t)\,\Delta t \tag{62}$$
$$= (0.34\text{ m/s})(1.5 - 0.5\text{ s}) + (1.02\text{ m/s})(2.5 - 1.5\text{ s}) + (1.70\text{ m/s})(3.5 - 2.5\text{ s}) \tag{63}$$
$$+ (2.38\text{ m/s})(4.5 - 3.5\text{ s}) + (3.06\text{ m/s})(5.5 - 4.5\text{ s}) + (3.74\text{ m/s})(6.5 - 5.5\text{ s}) \tag{64}$$
$$= 12.24\text{ m} \tag{65}$$

Numerical integration has a tendency to smooth out noise, so in general it is not as subject to the "noise" problem as numerical derivatives are. When using the rectangular rule, one may evaluate the function at the left edge of the horizontal (e.g. time) interval, at the right, edge, or at the center. There are other, more sophisticated, numerical integration methods that may give better results, such as the trapezoidal rule and Simpson's rule. You'll study these in a more comprehensive calculus course.

6 More Examples

Area of a Circle

You learned the formula for the area of a circle in elementary school: $A = \pi R^2$, where R is the radius of the circle. We can use integral calculus to derive this formula. The simplest way to approach this using rectangular coordinates is to find the area of a quarter circle and multiply by 4. Let's say the circle has radius R and center at the origin. Then the equation for the circle is

$$x^2 + y^2 = R^2 \tag{66}$$

or

$$y = \pm\sqrt{R^2 - x^2} \tag{67}$$

For the quarter circle in the first quadrant, we use only the $+$ sign, which corresponds to the upper semicircle:

$$y = \sqrt{R^2 - x^2} \tag{68}$$

as let x go from 0 to R to get the quarter-circle in the first quadrant. The area under this quarter-circle curve is then

$$\int_0^R \sqrt{R^2 - x^2}\,dx \tag{69}$$

This is a fairly complicated integral to work out. Often in cases like this, we consult a published table of integrals [1] to find the result already worked out for us. From a published table of integrals, we find the integral to be

$$\int_0^R \sqrt{R^2 - x^2}\,dx = \frac{1}{2}\left[x\sqrt{R^2 - x^2} + R^2 \tan^{-1}\left(\frac{x}{\sqrt{R^2 - x^2}}\right)\right]\Bigg|_0^R \tag{70}$$
$$= \frac{1}{2}\left(\frac{\pi}{2}R^2 - 0\right) = \frac{\pi}{4}R^2 \tag{71}$$

[1] Some well-known tables of integrals are found in the *CRC Standard Mathematical Tables and Formulae*; *Tables of Integrals and Other Mathematical Data* by Dwight; and the massive *Table of Integrals, Series, and Products* by Gradshteyn and Ryzhik.

The area of a circle is then 4 times this:

$$A = 4 \times \frac{\pi}{4} R^2 = \pi R^2 \tag{72}$$

and we have derived the famous formula $A = \pi R^2$.

It's actually simpler to work this problem in polar coordinates, although it leads to a *double integral*. Imagine a circle of radius R, whose center is at the origin. Now imagine a series of straight lines radiating away from the origin, and concentric circles around the origin, just as you have with polar graph paper. These lines divide the interior of the circle up into a series of little "boxes" with curved edges. If you make lots of lines, these boxes will be very small, and if they're infinitesimally small, you can treat them as rectangles. A general infinitesimal "rectangle" will have one side of length dr, and another of (arc) length $r\,d\theta$. The infinitesimal area of the little box is then the product of the lengths of the sides, $dA = r\,dr\,d\theta$. To get the area of a circle, we just add together the infinitesimal areas of all the little boxes inside the circle by integrating r from 0 to R, *and* integrating θ from 0 to 2π:

$$\text{area} = \int_0^{2\pi} \int_0^R dA = \int_0^{2\pi} \int_0^R r\,dr\,d\theta \tag{73}$$

This is called a *double integral*. The way to evaluate it is to evaluate the "inner" integral first, then make the result the integrand for the "outer" integral:

$$\int_0^{2\pi} \int_0^R r\,dr\,d\theta = \int_0^{2\pi} \left[\int_0^R r\,dr \right] d\theta \tag{74}$$

$$= \int_0^{2\pi} \left[\left. \frac{r^2}{2} \right|_0^R \right] d\theta \tag{75}$$

$$= \int_0^{2\pi} \left[\frac{R^2}{2} - \frac{0^2}{2} \right] d\theta \tag{76}$$

$$= \int_0^{2\pi} \left[\frac{R^2}{2} \right] d\theta \tag{77}$$

$$= \frac{R^2}{2} \int_0^{2\pi} d\theta \tag{78}$$

where in the last step we moved $R^2/2$ outside the integral because it's a constant. Now evaluate the θ integral:

$$\int_0^{2\pi} \int_0^R r\,dr\,d\theta = \frac{R^2}{2} \int_0^{2\pi} d\theta \tag{79}$$

$$= \left. \frac{R^2}{2} \theta \right|_0^{2\pi} \tag{80}$$

$$= \frac{R^2}{2} (2\pi - 0) \tag{81}$$

$$= \pi R^2 \tag{82}$$

And again we have derived the classical formula for the area of a circle.

Area of a Trapezoid

Suppose we have a trapezoid consisting of a side along the x axis, two parallel vertical sides at $x = 0$ and $x = h$, and a slanted top side that is a straight line. Let the vertical side at $x = 0$ have length a, and the vertical side at $x = h$ have length b. Then the classical formula for the area of a trapezoid is the mean of the lengths of the parallel sides times the distance between the parallel sides:

$$A = \frac{a+b}{2}h \tag{83}$$

Let's see if we can derive this formula from integral calculus. The slanted top side of the trapezoid passes through the points $(0, a)$ and (h, b). It therefore has equation

$$(y-a) = \frac{b-a}{h-0}(x-0) \tag{84}$$

or

$$y = \frac{b-a}{h}x + a \tag{85}$$

Using integral calculus, the area of the trapezoid is then the area under this line:

$$\int_0^h \left(\frac{b-a}{h}x + a\right) dx = \frac{b-a}{h}\int_0^h x\,dx + a\int_0^h dx \tag{86}$$

$$= \left(\frac{b-a}{h}\frac{x^2}{2} + ax\right)\Bigg|_0^h \tag{87}$$

$$= \left(\frac{b-a}{h}\frac{h^2}{2} + ah\right) - \left(\frac{b-a}{h}\frac{0^2}{2} + a(0)\right) \tag{88}$$

$$= h\left(\frac{b-a}{2} + a\right) \tag{89}$$

$$= h\left(\frac{b-a}{2} + \frac{2a}{2}\right) \tag{90}$$

$$= \frac{a+b}{2}h \tag{91}$$

and we have derived the classical formula.

Fence Enclosing Maximum Area

Let's look at an optimization problem. Say you have a pet dog, and want to make a rectangular fenced-in area in the back of your house for him to run around in. You get some fencing material, and plan to use the side of the house for one side of the play area, and the fencing material for the other three sides. Let's say you bought a total length L of fencing material, and let x be the length of the side of the play area that's along the side of the house. Now if $x = 0$, you'll have folded the fencing in half and set it perpendicular to the side of the house — you'll have a rectangle of size zero on one side, and therefore zero area. On the other hand, if $x = L$, then you'll have just set the fencing up against the house, and the play area will be a rectangle whose *other* side is size zero, and therefore encloses zero area again. Clearly there's some value of x in between 0 and L that must *maximize* the enclosed area. The question is: how do you maximize the total area of the play area? In other words, what must be the dimensions of the play area that maximizes the enclosed area for a given length of fencing L?

To solve this, we'll need to find a formula that gives the enclosed area as a function of x. Since x is the length of the side of the rectangle that's against the house, then the opposite side must also have length x; therefore the amount of fencing you have left over is $L - x$. This fencing will be used to make the other two sides, so each of the other sides of the rectangle will have length $(L-x)/2$. The rectangular play area will therefore be a rectangle whose sides parallel to the side of the house is x, and whose other sides have length $(L - x)/2$. The area of the rectangular play area is then

$$A(x) = x\,\frac{L-x}{2} = \frac{1}{2}(-x^2 + Lx) \tag{92}$$

This is the equation of a parabola opening downward, so it will have a peak that gives the maximum area. We can find the value of x at the peak (the maximum) because the slope of this curve is zero at the peak. All we need to do is compute the derivative (i.e. slope) of $A(x)$ with respect to x, then set that to zero.

$$\frac{d}{dx}\,A(x) = 0 \tag{93}$$

$$\frac{d}{dx}\,\left[\frac{1}{2}(-x^2 + Lx)\right] = 0 \tag{94}$$

$$\frac{1}{2}\,\frac{d}{dx}\,(-x^2 + Lx) = 0 \tag{95}$$

$$\frac{1}{2}\left[\frac{d}{dx}\,(-x^2) + \frac{d}{dx}\,(Lx)\right] = 0 \tag{96}$$

$$\frac{1}{2}\left[-\frac{d}{dx}\,x^2 + L\,\frac{d}{dx}\,x\right] = 0 \tag{97}$$

$$\frac{1}{2}\,[-2x + L] = 0 \tag{98}$$

$$-\,x + \frac{L}{2} = 0 \tag{99}$$

$$x = \frac{L}{2} \tag{100}$$

Therefore, to maximize the play area for your dog, you should make one side (the side parallel to the side of the house) equal to half the total amount of fencing ($L/2$); the remaining fencing will be divided equally among the other two sides, so the other sides will have length $L/4$. The total area enclosed — the maximum possible area for a length L of fencing — will be $(L/2)(L/4) = L^2/8$.

7 Main Ideas

We won't be doing anything very complicated with the calculus in this course; we'll leave mathematical rigor and more complicated problems to a dedicated calculus course. The the purposes of this course, here are the main ideas:

- The number dx is an *infinitesimal* number—a number on the x axis that is 'infinitely small," but not zero.

- The notation $\frac{d}{dx} f(x)$ (the *derivative*) gives the *slope* of the curve $f(x)$ at any x.
- As a special case, the notation $\frac{d}{dt} f(t)$ gives the *rate of change* of $f(t)$ with respect to time t.
- The notation $\int_a^b f(x)\,dx$ (the *integral*) gives the *area* under the curve $f(x)$ between $x = a$ and $x = b$.
- The derivative and integral are inverses of each other: $\frac{d}{dx} \int f(x)\,dx = f(x)$

8 Going Further

In this chapter we've only just touched on a few of the basic ideas behind the calculus. In a multi-semester course, you'll learn, among other things, how to derive the results presented here; about infinite series and sequences; how to take derivatives and integrals of more complex functions; advanced techniques; how to work in polar coordinates; how to work with functions of several variables; finding areas and volumes of solids of revolution; and how to solve differential equations.

An excellent and brief introduction to the calculus, at about the level of these notes, is *How to Enjoy Calculus* by Eli S. Pine. A typical college-level calculus textbook is *Calculus with Analytic Geometry* by Earl W. Swokowski.

9 Table of Derivatives

$$\frac{d}{dx}\, a = 0 \tag{101}$$

$$\frac{d}{dx}\, x = 1 \tag{102}$$

$$\frac{d}{dx}\, x^n = n x^{n-1} \tag{103}$$

$$\frac{d}{dx}\, \sqrt{x} = \frac{1}{2\sqrt{x}} \tag{104}$$

$$\frac{d}{dx}\, \sin x = \cos x \tag{105}$$

$$\frac{d}{dx}\, \cos x = -\sin x \tag{106}$$

$$\frac{d}{dx}\, \tan x = \sec^2 x \tag{107}$$

$$\frac{d}{dx}\, \sec x = \tan x \sec x \tag{108}$$

$$\frac{d}{dx}\, \csc x = -\cot x \csc x \tag{109}$$

$$\frac{d}{dx}\, \cot x = -\csc^2 x \tag{110}$$

$$\frac{d}{dx}\, e^x = e^x \tag{111}$$

$$\frac{d}{dx}\, \ln x = \frac{1}{x} \tag{112}$$

$$\frac{d}{dx}\, a^x = a^x \ln a \tag{113}$$

$$\frac{d}{dx}\, \log_a x = \frac{1}{x \ln a} \tag{114}$$

$$\frac{d}{dx}\, \sinh x = \cosh x \tag{115}$$

$$\frac{d}{dx}\, \cosh x = \sinh x \tag{116}$$

$$\frac{d}{dx}\, \tanh x = \operatorname{sech}^2 x \tag{117}$$

10 Table of Integrals

In the following table, an arbitrary constant C should be added to each result.

$$\int dx = x \tag{118}$$

$$\int a\,dx = ax \tag{119}$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1) \tag{120}$$

$$\int \sqrt{x}\,dx = \frac{2}{3}\sqrt{x^3} \tag{121}$$

$$\int \frac{1}{x}\,dx = \ln|x| \tag{122}$$

$$\int \sin x\,dx = -\cos x \tag{123}$$

$$\int \cos x\,dx = \sin x \tag{124}$$

$$\int \tan x\,dx = \ln|\sec x| \tag{125}$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| \tag{126}$$

$$\int \csc x\,dx = \ln|\csc x - \cot x| \tag{127}$$

$$\int \cot x\,dx = \ln|\sin x| \tag{128}$$

$$\int e^x\,dx = e^x \tag{129}$$

$$\int \ln x\,dx = x\ln x - x \tag{130}$$

$$\int a^x\,dx = \frac{a^x}{\ln a} \tag{131}$$

$$\int \log_a x\,dx = \frac{x\ln x - x}{\ln a} \tag{132}$$

$$\int \sinh x\,dx = \cosh x \tag{133}$$

$$\int \cosh x\,dx = \sinh x \tag{134}$$

$$\int \tanh x\,dx = \ln\cosh x \tag{135}$$