The Arithmetic-Geometric Mean

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Averages

Suppose you want to compute the *average* of a set of numbers. There are a number of ways of doing this; for example, if you arrange the numbers in order in a list, the value in the middle of the list is called the *median average*. You may have a set of numbers in which the same number occurs more than once; the value that occurs most often is called the *mode average*. The most common method, though, is to compute the sum of all the numbers and divide by how many numbers there are; this is called the *arithmetic mean*.

For two numbers x and y, the arithmetic mean a is given by

$$a = \frac{x+y}{2}. \tag{1}$$

Still another kind of average is called the *geometric mean*. The geometric mean g of two numbers x and y is the square root of their product:

$$g = \sqrt{xy}. \tag{2}$$

Arithmetic-Geometric Mean

A lesser-known type of average combines the arithmetic and geometric means together. Suppose we're given two numbers x and y. Then compute the arithmetic and geometric means of x and y:

$$a_1 = \frac{x+y}{2} \tag{3}$$

$$g_1 = \sqrt{xy}. \tag{4}$$

Now compute the arithmetic and geometric means *of these two means*:

$$a_2 = \frac{a_1+g_1}{2} \tag{5}$$

$$g_2 = \sqrt{a_1 g_1} \tag{6}$$

And now compute the arithmetic and geometric means of *these* two means:

$$a_3 = \frac{a_2+g_2}{2} \tag{7}$$

$$g_3 = \sqrt{a_2 g_2} \tag{8}$$

And so on. We iterate, repeating this process over and over,

$$a_{n+1} = \frac{a_n + g_n}{2} \tag{9}$$

$$g_{n+1} = \sqrt{a_n g_n} \tag{10}$$

and we discover that the two means will converge to the same number. The number to which these means converge is called the *arithmetic-geometric mean* of x and y.

Example

Suppose our two numbers are 7 and 12. Then their arithmetic mean is

$$a_1 = \frac{7+12}{2} = 9.5. \tag{11}$$

Their geometric mean is

$$g_1 = \sqrt{7 \times 12} = \sqrt{84} = 9.165151. \tag{12}$$

Now let's find the arithmetic-geometric mean of 7 and 12 by running these two means through the iteration process (Equations 9 and 10). We find

$$\begin{array}{ll} a_1 = 9.500000 & g_1 = 9.165151 \\ a_2 = 9.332576 & g_2 = 9.331074 \\ a_3 = 9.331074 & g_3 = 9.331074 \end{array}$$

So the two values have quickly converged to the arithmetic-geometric mean, which is 9.331074.

The Simple Plane Pendulum

Where would we use the arithmetic-geometric mean? One application is described in a recent paper by Adlaj [1], where it used to calculate the *exact* period of a simple plane pendulum of length L and amplitude θ_0. The appendices to the course notes give a formula for this period in terms of an infinite series:

$$T = 2\pi \sqrt{\frac{L}{g}} \left\{ 1 + \sum_{n=1}^{\infty} \left[\frac{(2n)!}{2^{2n}(n!)^2} \right]^2 \sin^{2n}\left(\frac{\theta_0}{2}\right) \right\} . \tag{13}$$

According to Adlaj, the exact period of a simple plane pendulum may be calculated more efficiently using the arithmetic-geometric mean, by means of the formula

$$T = 2\pi \sqrt{\frac{L}{g}} \times \frac{1}{\mathrm{agm}(1, \cos(\theta_0/2))} \tag{14}$$

where $\mathrm{agm}(x, y)$ denotes the arithmetic-geometric mean of x and y. The factor on the right involving the arithmetic-geometric mean is essentially a correction factor that corrects the small-angle approximation for the period ($T \approx 2\pi\sqrt{L/g}$) to the exact value.

Surprisingly, Eq. (14) is *not* just an approximation of Eq. (13). In theory, both formulæ return the same results, although Eq. (14) is actually numerically better behaved for amplitudes approaching $\theta_0 = 180°$.

Example

Suppose we have a simple plane pendulum with length $L = 1.000$ meter, and with amplitude $\theta_0 = 60°$. Using the series expansion (Eq. (13)) we compute the period $T = 2.153973$ sec.

Now find the period of the same pendulum again, but this time using the arithmetic-geometric mean, Eq. (14). We find $\text{agm}(1, \cos(60°/2)) = 0.931808$, which gives $T = 2.153973$ sec — the same result we got using the series expansion.

The arithmetic-geometric mean formula for the pendulum period (Eq. 14) has several advantages over the series expansion (Eq. 13):

- The formula is much simpler.
- It is computationally more efficient.
- It is numerically better behaved for amplitudes near $\theta_0 = 180°$.

References

[1] S. Adlaj, An Eloquent Formula for the Perimeter of an Ellipse. *Notices Amer. Math. Soc.*, **59**, 8, 1094 (September 2012).