Physics Recreations:
The Bikes and the Bee

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1 Introduction

In his book *Thinking Physics is Gedanken Physics (3rd ed.)*, author Lewis Carroll Epstein poses the following problem:

> Two bicyclists travel at a uniform speed of 10 mph toward each other. At the moment when they are 20 miles apart, a bumble bee flies from the front wheel of one of the bikes at a uniform speed of 25 mph directly to the wheel of the other bike. It touches it and turns around in a negligibly short time and returns at the same speed to the first bike, whereupon it touches the wheel and instantaneously turns around and repeats the back-and-forth trip over and over again — successive trips becoming shorter and shorter until the bikes collide and squash the unfortunate bee between the front wheels. What was the total mileage of the bee in its many back-and-forth trips from the time the bikes were 20 miles apart until its hapless end? (This can be very simple or very difficult, depending on your approach.)

As the author suggests, a bit of insight will give the correct answer immediately. But first we'll take a look at how to solve the problem the *hard way*.

2 Bike Positions

First, we will need formulæ for the positions of the bikes and the bee at any given time. Let's imagine the bikes and bee move along the x axis, with the origin halfway between the two bikes, so that they collide at the origin. Then one of the bikes (we'll call it Bike 1) is on the left, moving left to right in the $+x$ direction, with a speed of $+10$ mph and an initial position of $x = -10$ miles. Its x coordinate at any time t is then given by:

$$x_1(t) = 10t - 10. \tag{1}$$

(Here we assume all distances are in miles and times are in hours. Time $t = 0$ is at the start of the problem, when the bikes are 20 miles apart.) Similarly, the other bike (Bike 2) is on the right, moving right to left in the $-x$ direction, with a speed of -10 mph and an initial position of $x = +10$ miles. Its x coordinate at any time t is then given by:

$$x_2(t) = -10t + 10. \tag{2}$$

3 Bee Position

The position of the bee at any time is more complicated—we have to consider each segment of the bee's trip separately.

First Segment

During the first segment of its trip, the bee is moving, say, from Bike 2 to Bike 1, in the $-x$ direction, with a speed of -25 mph. Since its initial position was at $x = +10$, its position at any time during the first segment is

$$x_{b1}(t) = -25t + 10. \tag{3}$$

Now let's find the time t_1 at which the bee meets Bike 1. We set $x_1 = x_{b1}$ and find

$$\begin{align}
x_1(t) &= x_{b1}(t) \tag{4}\\
10t - 10 &= -25t + 10 \tag{5}\\
35t &= 20 \tag{6}\\
t_1 &= \tfrac{4}{7} \tag{7}
\end{align}$$

and so the position of the bee at the time it meets Bike 1 is

$$x_1\left(t_1 = \tfrac{4}{7}\right) = 10\left(\tfrac{4}{7}\right) - 10 = \tfrac{40}{7} - \tfrac{70}{7} = -\tfrac{30}{7}. \tag{8}$$

The distance traveled by the bee during the first segment is then

$$d_1 = |x_b(t_1) - x_b(t_0)| = \left|-\tfrac{30}{7} - 10\right| = \tfrac{100}{7} \tag{9}$$

Second Segment

During the second segment of its trip, the bee is moving from Bike 1 to Bike 2, in the $+x$ direction, with a speed of $+25$ mph. Since its initial position was at $x_1 = \frac{30}{7}$, its position at any time during the second segment is

$$x_{b2}(t) = 25\left(t - \tfrac{4}{7}\right) - \tfrac{30}{7} = 25t - \tfrac{130}{7}. \tag{10}$$

Here we had to subtract $\frac{4}{7}$ from t in order to keep t indexed so that $t = 0$ at the beginning of the problem, and the first encounter with a bike occurs at $t_1 = \frac{4}{7}$ and $x_1 = -\frac{30}{7}$. Now let's find the time t_2 at which the bee meets Bike 2. We set $x_2 = x_{b2}$ and find

$$\begin{align}
x_2(t) &= x_{b2}(t) \tag{11}\\
-10t + 10 &= 25t - \tfrac{130}{7} \tag{12}\\
35t &= \tfrac{200}{7} \tag{13}\\
t_2 &= \tfrac{200}{245} = \tfrac{40}{49} \tag{14}
\end{align}$$

and so the position of the bee at the time it meets Bike 2 is

$$x_2\left(t_2 = \tfrac{40}{49}\right) = -10\left(\tfrac{40}{49}\right) + 10 = -\tfrac{400}{49} + \tfrac{490}{49} = \tfrac{90}{49}. \tag{15}$$

The distance traveled by the bee during the first segment is then

$$d_2 = |x_b(t_2) - x_b(t_1)| = \left|\tfrac{90}{49} - \left(-\tfrac{30}{7}\right)\right| = \tfrac{300}{49}. \tag{16}$$

Third Segment

During the third segment of its trip, the bee is moving from Bike 2 to Bike 1, in the $-x$ direction, with a speed of -25 mph. Since its initial position was at $x_2 = \frac{90}{49}$, its position at any time during the second segment is

$$x_{b3}(t) = -25\left(t - \tfrac{40}{49}\right) + \tfrac{90}{49} = -25t + \tfrac{1090}{49}. \tag{17}$$

Once again we had to subtract an offset ($\frac{40}{49}$) from t in order to keep t indexed so that $t = 0$ at the beginning of the problem. Now let's find the time t_3 at which the bee meets Bike 1. We set $x_1 = x_{b3}$ and find

$$\begin{aligned} x_1(t) &= x_{b3}(t) && (18)\\ 10t - 10 &= -25t + \tfrac{1090}{49} && (19)\\ 35t &= \tfrac{1580}{49} && (20)\\ t_3 &= \tfrac{1580}{1715} = \tfrac{316}{343} && (21) \end{aligned}$$

and so the position of the bee at the time it meets Bike 1 is

$$x_1\left(t_3 = \tfrac{316}{343}\right) = 10\left(\tfrac{316}{343}\right) - 10 = -\tfrac{3160}{343} - \tfrac{3430}{343} = -\tfrac{270}{343}. \tag{22}$$

The distance traveled by the bee during the first segment is then

$$d_3 = |x_b(t_3) - x_b(t_2)| = \left|-\tfrac{270}{343} - \tfrac{90}{49}\right| = \tfrac{900}{343}. \tag{23}$$

nth Segment

Let's look at the distances we've computed so far:

$$\begin{aligned} d_1 &= \tfrac{100}{7}\\ d_2 &= \tfrac{300}{49}\\ d_3 &= \tfrac{900}{343} \end{aligned}$$

By now, we can see a pattern: the numerators are 100 times powers of 3, and the denominators are powers of 7. Apparently, then, the distance traveled by the bee during the nth segment is

$$d_n = \frac{100 \times 3^{n-1}}{7^n} = \frac{100}{3}\left(\frac{3}{7}\right)^n. \tag{24}$$

The total distance d traveled by the bee is then

$$\sum_{n=1}^{\infty} d_n = \sum_{n=1}^{\infty} \frac{100}{3}\left(\frac{3}{7}\right)^n = \frac{100}{3}\sum_{n=1}^{\infty}\left(\frac{3}{7}\right)^n, \tag{25}$$

which we recognize as a geometric series. Let's change the lower summation index from 1 to 0:

$$d = \frac{100}{3}\left[\sum_{n=0}^{\infty}\left(\frac{3}{7}\right)^{n} - 1\right]. \tag{26}$$

Now from the theory of geometric series, we know that

$$\sum_{n=0}^{\infty} r^{n} = \frac{1}{1-r}, \tag{27}$$

provided $|r| < 1$. Here $r = \frac{3}{7}$, so we have

$$d = \frac{100}{3}\left(\frac{1}{1-\frac{3}{7}} - 1\right) \tag{28}$$

$$= \frac{100}{3}\left(\frac{7}{4} - 1\right) \tag{29}$$

$$= \frac{100}{3} \times \frac{3}{4} \tag{30}$$

$$= \boxed{25 \text{ miles}} \tag{31}$$

4 The Easy Way

Here's the easy solution: the bikes collide after 1 hour. During that 1 hour, the bee is constantly moving at a speed of 25 mph. Therefore, the bee travels 25 miles.