Physics Recreations:
Leap Seconds

Dr. D.G. Simpson
Department of Physical Sciences and Engineering
Prince George's Community College

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1 Introduction

Most people are familiar with the concept of *leap years*: every year that is divisible by 4 is a leap year, during which we add an extra day, February 29. (Less well known is the actual leap-year rule, which is more complicated: a year is a leap year if it's divisible by 4, but *not* if it's divisible by 100, unless it's also divisible by 400. Hence 1800, 1900, and 2100 are *not* leap years, but 2000 and 2400 *are* leap years because they're divisible by 400.)

A similar, but less publicized, idea is the *leap second*. During a leap second, an extra second is added to the clock just before midnight. When a leap second is introduced, a clock will read (p.m.):

...
11:59:57
11:59:58
11:59:59
11:59:60
12:00:00
12:00:01
...

The time called 11:59:60 is the leap second. Its introduction has the same effect as turning the clock back by one second.

Leap seconds occur at irregular intervals, but come (very roughly) every 18 months or so. The decision to introduce a leap second is made by the International Earth Rotation Service a few month in advance. When they occur, leap seconds are introduced just before midnight on either June 30 or December 31. So far, leap seconds have been added on the dates shown in Table 1.

Table 1. Leap Seconds to Date.

Dec. 31, 1971	June 30, 1985
June 30, 1972	Dec. 31, 1987
Dec. 31, 1972	Dec. 31, 1989
Dec. 31, 1973	Dec. 31, 1990
Dec. 31, 1974	June 30, 1992
Dec. 31, 1975	June 30, 1993
Dec. 31, 1976	June 30, 1994
Dec. 31, 1977	Dec. 31, 1995
Dec. 31, 1978	June 30, 1997
Dec. 31, 1979	Dec. 31, 1998
June 30, 1981	Dec. 31, 2005
June 30, 1982	Dec. 31, 2008
June 30, 1983	

2 The Reason for Leap Seconds

Why do we have leap seconds? It's because we don't actually keep time according to the Earth's rotation: we don't divide the day into 24 equal parts to make hours, then subdivide those into minutes and seconds. Instead, we keep time by extremely accurate atomic clocks, which are much more accurate than the Earth's rotation. Over long periods of time, we find that the Earth's rate of rotation (its angular velocity) is gradually slowing. Introducing leap seconds lets us set our clocks back by one second from time to time, in order to keep our clocks synchronized with the Earth's rotation.

So why is the Earth's angular velocity slowing, causing our days to become gradually longer? It's because the Moon exerts a gravitational pull on the Earth's oceans (creating tides), and the friction between the Earth and its oceans causes the Earth to very gradually slow its rotation rate.

But there's a problem here. Recall that the angular momentum of a solid body is given by

$$L = I\omega, \tag{1}$$

where L is the angular momentum, I is the moment of inertia, and ω is the angular velocity. If the Earth's angular velocity is gradually decreasing and its moment of inertia is constant, then that must mean that its angular momentum is decreasing with time. Yet angular momentum is a conserved quantity; where does the angular momentum lost by the Earth go?

The answer is: it is transferred to the Moon. As the Earth loses *rotational* angular momentum, it is transferred to the Moon in the form of extra *orbital* angular momentum gained by the Moon. As the Earth slows its rotation and loses angular momentum, the Moon moves farther away from the Earth, gaining the same amount of angular momentum. The total angular momentum of the Earth-Moon system remains constant.

Let's first look at what happens to the Earth. Quantitatively, the Earth loses about 1.5×10^{24} N m s of rotational angular momentum each year, and this same amount is gained by the Moon each year as extra orbital angular momentum. The Earth's rotational angular momentum L_e is given by

$$L_e = I_e\,\omega = \frac{2\pi I_e}{T}, \tag{2}$$

where I_e is the moment of inertia of the Earth about its rotation axis, and T is the period of rotation (1 day). Differentiating gives the relationship between the change in rotation period ΔT for a given change in angular momentum ΔL_e:

$$\Delta L_e = \frac{2\pi I_e}{T^2}\,\Delta T. \tag{3}$$

Given the annual loss of angular momentum $\Delta L_e = 1.5 \times 10^{24}$ N m s, and that the moment of inertia of the Earth is $I_e = 8.04 \times 10^{37}$ kg m^2, this gives an annual increase in the length of the day of $\Delta T = 20\ \mu$s. Leap seconds are introduced partly because of this gradual increase in the length of the day, and partly because 1 second defined by atomic clocks is slightly shorter than 1 second would be if based on the Earth's rotation: our atomic clocks run slightly too fast, so we have to set them back occasionally.

Now let's see what happens to the Moon. The Moon's orbital angular momentum L_m is given by

$$L_m = M_m v R, \tag{4}$$

where M_m is the mass of the Moon, v is its orbital speed, and R is its orbital radius. The orbital speed v can be found by setting the gravitational force equal to the centripetal force:

$$\frac{G M_e M_m}{R^2} = \frac{M_m v^2}{R}. \tag{5}$$

Solving for v gives

$$v = \sqrt{\frac{G M_e}{R}}. \tag{6}$$

Now substituting this expression for v into Eq. (4), we find

$$\begin{aligned} L_m &= M_m v R && (7)\\ &= M_m \sqrt{\frac{G M_e}{R}}\, R && (8)\\ &= M_m \sqrt{G M_e R}. && (9) \end{aligned}$$

As the Moon moves into a radius of larger radius, its orbital radius R will increase (causing L_m to increase), but its orbital speed v will decrease (causing L_m to decrease). Overall, increasing R increases the orbital angular momentum more than the decrease in v decreases it, and the orbital angular momentum will increase as $\sqrt{R}$, as shown by Eq. (9).

Differentiating Eq. (9) shows how the orbital radius R changes for a given change in angular momentum:

$$\Delta L_m = \frac{1}{2} M_m \sqrt{\frac{G M_e}{R}}\,\Delta R. \tag{10}$$

Given the annual increase in orbital angular momentum $\Delta L_m = 1.5 \times 10^{24}$ N m s, the mass of the Moon $M_m = 7.349 \times 10^{22}$ kg, the Earth gravitational constant $G M_e = 3.986005 \times 10^{14}$ m^3 s^{-2}, and the Moon's orbital radius $R = 3.844 \times 10^8$ m, we find an annual increase in the Moon's orbital radius of $\Delta R = 4$ cm. This means that each year, the Moon moves 4 cm farther from the Earth. This has been confirmed by laser ranging measurements made using retroreflectors left on the Moon by the *Apollo* astronauts.

3 The Changing Earth-Moon System

All this means that in the distant past, the days were shorter and the Moon was closer to the Earth. During the time of the dinosaurs, for example, the day would have been around 23 hours long. In the distant future, the length of the day will gradually increase, and the Moon will move farther from the Earth.

As you know, the Moon is “tidally locked” to the Earth: it only presents one face toward the Earth, which we call the “near side” of the Moon. As the Moon moves farther from the Earth, the Earth-Moon system will eventually reach a state where the Earth also becomes tidally locked to the Moon—in other words, the Earth will only present one face to the Moon, and the Earth and Moon will be tidally locked to each other. At that point, the orbital period of the Moon and the length of the day will be equal (over 1000 hours), and the Moon will be stationary in the sky. From some parts of the Earth, the Moon will never be visible; from other parts of the Earth, the Moon will be visible (and much smaller than today), but will never change its position in the sky. At that point, the Moon’s recession from the Earth will cease.

However, it is expected that this won’t happen until some 50 billion years in the future. (By comparison, the current age of the Universe is 13.7 billion years.) Long before then—about 5 billion years from now—the Sun will run out of its hydrogen fuel and expand into a red giant star, perhaps engulfing and vaporizing the Earth and Moon.