Physics Recreations: Bicycle Gearing

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If you ride a bicycle, your bike probably has several gears on it. Most bikes have five gears in the rear and two or three gears in front, leading them to be called "ten-speed" or "fifteen-speed" bikes. These so-called "speeds" are really just different combinations of gears; the number of gear combinations is the product of the number of front gears and the number of rear gears.

Many people ride bikes by just randomly shifting the front and rear gears until they find a combination that seems about right. But by using some simple physics and mathematics to understand how the gearing is set up on your bike, you can devise an efficient shifting strategy that will allow you to systematically shift from low gear to high gear and back again, without fumbling and guessing at which combination of gears is best.

1 Data Collection

You will need two collect two sets of data for your bicycle:

- Your wheel diameter (in inches); and
- The number of teeth in each gear (both front and rear sets of gears).

The front set of gears is called the *chainring*, while the rear set of gears is called the *cog*. I'll use my ten-speed bicycle as an example. My bicycle has wheels that are 27 inches in diameter. For the number of teeth in each gear, I count:

- Front gears (chainring): 44 and 52 teeth;
- Rear gears (cog): 14, 17, 20, 24, and 28 teeth.

This is all the data that is needed.

2 Gearing Calculations and Shifting Strategies

Now we can begin the calculations. We want to set up a table showing each combination of gears on the bicycle. For each combination, we will calculate the *gear size* (in units of *gear-inches*) for that gearing combination; this is defined as the product of the wheel diameter (in inches) and the *gear ratio* F/R:

$$\boxed{G = D\,\frac{F}{R}} \tag{1}$$

Here G is the gear size in gear-inches, D is the wheel diameter in inches, F is the number of teeth in the front (chainring) gear, and R is the number of teeth in the rear (cog) gear. Physically, G is the diameter of an equivalent wheel on a gearless bicycle; for example, using a 100-inch gear is equivalent to having 100-inch diameter wheels. Low values of G are low gears: you can turn the pedals many times and the bicycle won't move very far, which is helpful for going up steep hills. High values of G are high gears: the bike will move a long way with relatively little turning of the pedals, which is helpful for going fast on level ground or downhill.

For my bicycle, I find for the gear sizes G for each gear combination, using equation (1) (and rounding to the nearest gear-inch):

Table 1. Gear sizes G (gear-in) for my bicycle.

	Chainring (F)	
Cog (R)	44	52
14	85	100
17	70	83
20	59	70
24	50	59
28	42	50

As you can see, although my bike is technically a "ten-speed", there are several duplications in the table. For example, I can get a 50-inch gear by using either the 44-tooth gear in front and the 24-tooth gear in back, *or* by using the 52-tooth gear in front and the 28-tooth gear in back. There are similar duplications for the 59- and 70-inch gears, and the 83-inch and 85-inch gears are nearly the same. So although I have a "ten-speed" bike, I really only have six different gear sizes: 42, 50, 59, 70, 83, and 100 gear-inches.

Now I'll examine the above table and try to devise a shifting strategy that will allow me to cycle through these six gear sizes in some reasonable way, without having to move the shifting levers too much. One possibility, for example, would be to use the small (44-tooth) front gear only for the lowest gear (42 gear-inches), then switch to the large 52-tooth front gear and stay there for all the other gearings, changing only the rear gear to get 50, 59, 70, 83, and 100 gear-inches. I prefer something a bit more symmetrical, though. The table below shows my preferred shifting strategy, where I've struck out the gearings I don't use (all of which are duplicates anyway, so I don't need them):

Table 2. Shifting strategy for my bicycle.

	Chainring (F)	
Cog (R)	44	52
14	~~85~~	100
17	~~70~~	83
20	59	70
24	50	~~59~~
28	42	~~50~~

As this table shows, my shifting strategy is to use my small (44-tooth) front gear for 42, 50, and 59 gear-inches; then I change to the large (52-tooth) front gear to get 70, 83, and 100 gear-inches.

3 Cadence and Shifting Speed

Another question you might like to answer is: at what speed should you change gears? This is related to the rider's *cadence*, which is the number of revolutions of the pedals the rider makes per unit time. A typical cadence used by experienced riders is about 80 revolutions per minute.

Now let's see what bicycle speed corresponds to a cadence of, say, 80 revolutions per minute for each of the gear sizes found in the tables above. We know from elementary mechanics that the relation between the wheel linear speed (and therefore also the bicycle speed) v and the wheel angular velocity Ω is given by

$$v = r\Omega, \tag{2}$$

where r is the wheel radius. Now the wheel angular velocity Ω is related to the pedal angular velocity ω (the cadence) through the gear ratio F/R:

$$\Omega = \omega \, \frac{F}{R}, \tag{3}$$

where F and R are the number of teeth in the front and rear gears, as before. Combining equations (2) and (3), we get

$$v = r\omega \, \frac{F}{R}. \tag{4}$$

In terms of the wheel diameter $D = 2r$, this becomes

$$v = \frac{D}{2} \, \omega \, \frac{F}{R}. \tag{5}$$

Now let's deal with units. It's customary to have the bike speed v in miles per hour (mph), the wheel diameter D in inches, and the rider's cadence ω in revolutions per minute (rpm). Angular units in equation (5) are implicitly in radians, so we'll have to convert to that from revolutions. Doing the appropriate unit conversions:

$$\begin{aligned} v &= \left(\frac{D \text{ in}}{2}\right) \left(\omega \frac{\text{rev}}{\text{min}}\right) \left(\frac{F}{R}\right) \times \frac{1 \text{ ft}}{12 \text{ in}} \times \frac{1 \text{ mi}}{5280 \text{ ft}} \times \frac{2\pi \text{ rad}}{1 \text{ rev}} \times \frac{60 \text{ min}}{1 \text{ hr}} && (6) \\ &= \frac{\pi}{1056} \, D \, \omega \, \frac{F}{R} && (7) \end{aligned}$$

By equation (1), $G = D(F/R)$; making this substitution into equation (7) gives

$$\boxed{v = \frac{\pi}{1056}\, G\, \omega} \tag{8}$$

In equation (8), v is the bicycle speed in mph, G is the gear size in gear-inches, and ω is the rider's cadence in rpm. For example, for each of the six gear sizes G on my bicycle, I can compute the speed v that corresponds to a comfortable cadence of $\omega =$ 80 rpm:

Table 3. Speeds corresponding to $\omega = 80$ rpm for my bicycle.

G (gear-in)	v (mph)
42	10
50	12
59	14
70	17
83	20
100	24

Using this table and my bicycle's speedometer, I can estimate at roughly what speed I should expect to shift from one gear size to the next to maintain a comfortable cadence of about 80 rpm.

I've actually used these results on number of long bicycle rides. (I had to memorize the table, of course.) In practice, I typically use these as the nominal "cruising speeds" for each gear size. For example, if I'm using my 59-inch gear at 14 mph and want to go a little faster, I'll speed up to 17 mph and shift up to the 70-inch gear. I use the highest (100-inch) gear for any speed over about 23 mph. (I've gone as fast as 40 mph in this gear.)

4 Your Bicycle

The numbers shown here are just an example, and are specific to my bicycle. To do this for *your* bicycle, find your wheel diameter D (in inches), and count how many teeth are on each of your chainring and cog gears. Then using equation (1), compute the gear sizes G (in gear-inches) for each gear combination; you will then have a table like Table 1. By studying your table of gear sizes, you can figure out a convenient gear-shifting sequence for moving from one gear size G to the next, while avoiding duplicate gear sizes. Equation (8) allows you to calculate the bike speed v for each gear size G for a fixed cadence ω, which gives you an idea of when to shift from one gear size to the next (assuming your bike is equipped with a speedometer for measuring your speed v).

(Reference: *Bicycle Gearing: A Practical Guide* by Dick Marr, Mountaineer Books, 1989.)